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ON DOMINANCE RELATIONS IN DISJUNCTIVE SET-VALUED ORDERED INFORMATION SYSTEMS

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Set-valued information systems are generalized models of single-valued information systems. Its semantic interpretation can be classified into two categories: disjunctive and conjunctive. We focus on the former in this paper. By introducing four types of dominance relations to the disjunctive set-valued information systems, we establish a dominance-based rough sets approach, which is mainly based on the substitution of the indiscernibility relation by the dominance relations. Furthermore, we develop a new approach to sorting for objects in disjunctive set-valued ordered information systems, which is based on the dominance class of an object induced by a dominance relation. Finally, we propose criterion reductions of disjunctive set-valued ordered information systems that eliminate only those information that are not essential from the ordering of objects. The approaches show how to simplify a disjunctive set-valued ordered information system. Throughout this paper, we establish in detail the interrelationships among the four types of dominance relations, which include corresponding dominance classes, rough sets approaches, sorting for objects and criterion reductions. These results give a kind of feasible approaches to intelligent decision making in disjunctive set-valued ordered information systems.

Keywords: Information systems; dominance relation; rough sets; decision-making; criterion reduction.

1. Introduction

Data mining and knowledge management are very important research issues in management science field.[1,2] In these issues, one often encounters various types of data. Rough set theory, introduced by Pawlak,[3,4] has been conceived as a tool to conceptualize and analyze various types of data. It can be used in the attribute-value representation model to describe the dependencies among attributes and evaluate the significance of attributes and derive decision rules. It has important applications to intelligence decision and cognitive sciences, as a tool to deal with vagueness and uncertainty of facts, and in classification.[5–13]

Rough-set-based data analysis starts from a data table, called information systems. The information systems contains data about objects of interest, characterized by a finite set of attributes.[14–20] It is often interesting to discover some dependency relationships (patterns) among attributes.

The original rough sets theory does not consider attributes with preference-ordered domains, that is, criteria. However, in many real situations, we are often faced with the problems in which the ordering of properties of the considered attributes plays a crucial role. One such type of problem is the ordering of objects. For this reason, Greco, Matarazzo, and Slowinski[21–24] proposed an extension of rough set theory, called the dominance-based rough sets approach (DRSA) to take into account the ordering properties of criteria. This innovation is mainly based on substitution of the indiscernibility relation by a dominance relation. In DRSA, where condition attributes are criteria and classes are preference ordered, the knowledge approximated is a collection of upward and downward unions of classes and the granules of knowledge are sets of objects defined by using a dominance relation. In recent years, many studies have been made about DRSA.[25–28]

Set-valued information systems are an important type of data tables, and generalized models of single-valued information systems. Let U be a finite set of objects, called the universe of discourse and AT be a finite set of attributes. With every attribute $a \in AT$, a set of its values V_a is associated. Then, $f : U \times AT \to V$ is a total function such that $f(x, a) \subseteq V_a$ for every $a \in AT, x \in U$. If each attribute has a unique attribute value, then (U, AT, V, f) with $V = \bigcup_{a \in AT} V_a$ is called a single-valued information system; if a system is not a single-valued information system, it is called a set-valued (multi-valued) information system. If the attributes only have two types property, i.e. condition and decision attributes, then such an information system is called a set-valued decision information system. A set-valued decision information system is always denoted by $S = (U, C \cup \{d\}, V, f)$, where C is a finite set of condition attributes, d is a decision attribute with $C \cap d = \emptyset$.

There are many ways to give a semantic interpretation of the set-valued information systems,[29–32] here we summarize them as two types[33]:

Type I: For $x \in U$ and $c \in C$, $c(x)$ is interpreted disjunctively. For example: If c is the attribute "speaking a language", the $c(x) = \{\text{German, Polish, France}\}$ can be interpreted as: x speaks German, Polish, or France, and x can speak only one of them. Incomplete information systems with some unknown attribute values or partial known attribute values[16,34,35] are such types of set-valued information systems. Under the consideration, we call it a *disjunctive set-valued information system.*

Type II: For $x \in U$ and $c \in C$, $c(x)$ is interpreted conjunctively. For example: If c is the attribute "speaking a language", then $c(x) = \{\text{German, Polish, France}\}$ can be interpreted as: x speaks German, Polish, and France. When considering the attribute "feeding habits" of animals, if we denote the attribute value of herbivore as "0" and carnivore as "1", then animals possessing attribute value $\{0, 1\}$ are considered as possessing both herbivorous and carnivorous nature. Let us take blood origin for another example, if we denote the three types of pure blood as "0", "1" and "2", then we can denote the mixed-blood as $\{0, 1\}$ or $\{1, 2\}$, etc. Under the interpretation, we say it to be a *conjunctive set-valued information system.*

In this paper, we focus on disjunctive set-valued information systems. The main objective of this article is to introduce four dominance relations to a disjunctive set-valued information system according to the relation between set-values, and establish the relationships among these dominance relations, their rough sets and decision makings induced by them.

The rest of this paper is organized as follows. Some preliminary concepts about ordered information systems is briefly recalled in Sec. 2. In Sec. 3, we introduce four dominance relations, such as up dominance relation, down dominance relation, up-down dominance relation and down-up dominance relation, to a disjunctive set-valued information system, and establish the relationship among them as well. In Sec. 4, for the four dominance relations, we establish the dominance-based rough sets approaches, and also analyze the differences between them. In Sec. 5, we investigate sorting problem for all objects on the universe in decision-making by using the four dominance relations proposed in disjunctive set-valued information systems. In Sec. 6, we present the approaches to the criterion reductions of a disjunctive set-valued ordered information system by using the discernibility matrices, and establish the interrelationship among the four types of criterion reductions as well. In Sec. 7, through a venture-investment issue, it is illustrated that how to make a decision by using the approaches proposed in this paper. Finally, we conclude the paper with a summary in Sec. 8.

2. Ordered Information Systems

In this section, we briefly review some basic concepts of ordered information systems and set-valued information systems.

An information system (IS) is an quadruple $S = (U, AT, V, f)$, where U is a finite nonempty set of objects and AT is a finite nonempty set of attributes, $V = \bigcup_{a \in AT} V_a$ and V_a is a domain of attribute a, $f : U \times AT \to V$ is a total function such that $f(x, a) \in V_a$ for every $a \in AT$, $x \in U$, called an information function.[36] A decision table is a special case of an information system in which, among the attributes, we distinguish one called a decision attribute. The other attributes are called condition attributes. Therefore, $S = (U, C \cup d, V, f)$ and $C \cap d = \emptyset$,[36] where set C contains so-called condition attributes and d, the decision attribute.

If the domain (scale) of a condition attribute is ordered according to a decreasing or increasing preference, then the attribute is a criterion.[37–40]

Definition 2.1. An information system is called an ordered information system (OIS) if all condition attributes are criterions.[41]

It is assumed that the domain of a criterion $a \in AT$ is completely pre-ordered by an outranking relation $\succcurlyeq_a$; $x \succcurlyeq_a y$ means that x is at least as good as (outranks) y with respect to criterion a. In the following, without any loss of generality, we consider a condition criterion having a numerical domain, that is, $V_a \subseteq \mathbf{R}$ ($\mathbf{R}$ denotes the set of real numbers) and being of type gain, that is, $x \succcurlyeq_a y \Leftrightarrow f(x, a) \geq f(y, a)$ (according to increasing preference) or $x \succcurlyeq_a y \Leftrightarrow f(x, a) \leq f(y, a)$ (according to decreasing preference), where $a \in AT$, $x, y \in U$. For a subset of attributes $B \subseteq C$, we define $x \succcurlyeq_B y \Leftrightarrow \forall a \in B, f(x, a) \geq f(y, a)$. In other words, x is at least as good as y with respect to all attributes in B. In general, the domain of the condition criterion may be also discrete, but the preference order between its values has to be provided.

In the following, we review the dominance relation that identifies granules of knowledge. In a given OIS, we say that x dominates y with respect to $B \subseteq C$ if $x \succcurlyeq_B y$, and denoted by $xR_B^{\geqq}y$. That is

$$R_B^{\geqq} = \{(y, x) \in U \times U \mid y \succcurlyeq_B x\}. \tag{2.1}$$

Obviously, if $(y, x) \in R_B^{\geqq}$, then y dominates x with respect to B.

Let B_1 be attributes set according to increasing preference, B_2 attributes set according to decreasing preference, hence $B = B_1 \cup B_2$. The granules of knowledge induced by the dominance relation $R_B^{\geqq}$ are the set of objects dominating x, i.e.

$$\begin{aligned}[x]_B^{\geqq} &= \{y \in U \mid f(y, a_1) \geq f(x, a_1)(\forall a_1 \in B_1), f(y, a_2) \leq f(x, a_2)(\forall a_2 \in B_2)\} \\ &= \{y \in U \mid (y, x) \in R_B^{\geqq}\}\end{aligned}$$

and the set of objects dominated by x,

$$\begin{aligned}[x]_B^{\leqq} &= \{y \in U \mid f(y, a_1) \leq f(x, a_1)(\forall a_1 \in B_1), f(y, a_2) \geq f(x, a_2)(\forall a_2 \in B_2)\} \\ &= \{y \in U \mid (x, y) \in R_B^{\geqq}\},\end{aligned}$$

which are called the B-dominating set and the B-dominated set with respect to $x \in U$, respectively.

Let $U/R_B^{\geqq}$ denote classification on the universe, which is the family set $\{[x]_B^{\geqq} \mid x \in U\}$. Any element from $U/R_B^{\geqq}$ will be called a dominance class with respect to B. Dominance classes in $U/R_B^{\geqq}$ do not constitute a partition of U in general. They constitute a covering of U.

For simplicity, without any loss of generality, in the following we only consider condition attributes with increasing preference.

The following property can be easily concluded.[28]

Theorem 2.1. *Let $R_B^{\geqslant}$ be a dominance relation, then*

(1) *$R_B^{\geqslant}$ is reflexive, transitive and unsymmetric, so it is not an equivalence relation;*
(2) *if $A \subseteq B \subseteq C$, then $R_C^{\geqslant} \subseteq R_B^{\geqslant} \subseteq R_A^{\geqslant}$;*
(3) *if $A \subseteq B \subseteq C$, then $[x]_C^{\geqslant} \subseteq [x]_B^{\geqslant} \subseteq [x]_A^{\geqslant}$;*
(4) *if $x_j \in [x_i]_B^{\geqslant}$, then $[x_j]_B^{\geqslant} \subseteq [x_i]_B^{\geqslant}$ and $[x_i]_B^{\geqslant} = \bigcup\{[x_j]_B^{\geqslant} : x_j \in [x_i]_B^{\geqslant}\}$;*
(5) *$[x_i]_B^{\geqslant} = [x_j]_B^{\geqslant}$ iff $f(x_i, a) = f(x_j, a)(\forall a \in B)$;*
(6) *$F = \{[x]_B^{\geqslant} \mid x \in U\}$ constitutes a covering of U.*

Example 2.1. An OIS is presented in Table 1, where $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $AT = \{a_1, a_2, a_3\}$.

The dominance classes determined by AT are

$$[x_1]_{AT}^{\geqslant} = \{x_1, x_2, x_5, x_6\}, \quad [x_2]_{AT}^{\geqslant} = \{x_2, x_5, x_6\}, \quad [x_3]_{AT}^{\geqslant} = \{x_2, x_3, x_4, x_5, x_6\},$$
$$[x_4]_{AT}^{\geqslant} = \{x_4, x_6\}, \quad [x_5]_{AT}^{\geqslant} = \{x_5\}, \quad [x_6]_{AT}^{\geqslant} = \{x_6\}.$$

3. Dominance Relations in Disjunctive Set-Valued Ordered Information Systems

However, it may happen that some of the attribute values for an object are set-valued in practical issues. Therefore, a so-called set-valued information system, is usually used to indicate such a situation.

Let $S = (U, AT, V, f)$ be a set-valued information system, where U is a non-empty finite set of objects; AT is a finite set of attributes, V is the set of attributes values and f is a mapping from $U \times AT$ to V such that $f : U \times AT \to 2^V$ is a set-valued mapping. In this situation, the cardinality $|f(x, a)| \geq 1$, $\forall x \in U, a \in AT$. The following example presents a set-valued information system.

Example 3.1. A set-valued information system is presented in Table 2, where $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$, $AT = \{a_1, a_2, a_3, a_4, a_5\}$.

For a disjunctive set-valued information system $S = (U, AT, V, f)$, the relationships among any set $f(x, a)$, $x \in U, a \in AT$ are disjunctive. In decision-making,

Table 1. An ordered information system.

U	a_1	a_2	a_3
x_1	1	2	1
x_2	3	2	2
x_3	1	1	2
x_4	2	1	3
x_5	3	3	2
x_6	3	2	3

Table 2. A set-valued information system.[33]

U	a_1	a_2	a_3	a_4	a_5
x_1	$\{1\}$	$\{0,1\}$	$\{0\}$	$\{1,2\}$	$\{2\}$
x_2	$\{0,1\}$	$\{2\}$	$\{1,2\}$	$\{0\}$	$\{0\}$
x_3	$\{0\}$	$\{1,2\}$	$\{1\}$	$\{0,1\}$	$\{0\}$
x_4	$\{0\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{0,2\}$
x_5	$\{2\}$	$\{1\}$	$\{0,1\}$	$\{0\}$	$\{1\}$
x_6	$\{0,2\}$	$\{1\}$	$\{0,1\}$	$\{0\}$	$\{1\}$
x_7	$\{1\}$	$\{0,2\}$	$\{0,1\}$	$\{1\}$	$\{2\}$
x_8	$\{0\}$	$\{2\}$	$\{1\}$	$\{0\}$	$\{0,1\}$
x_9	$\{1\}$	$\{0,1\}$	$\{0,2\}$	$\{1\}$	$\{2\}$
x_{10}	$\{1\}$	$\{1\}$	$\{2\}$	$\{0,1\}$	$\{2\}$

we always consider a binary dominance relation between objects that are possibly dominant in terms of values of attributes set A in disjunctive set-valued information systems. Under this consideration, we call S a *disjunctive set-valued ordered information system*.

Let $S = (U, AT, V, f)$ be a disjunctive set-valued information system, $A \subseteq AT$. In the following, let us consider four possible dominance relations between objects as follows:

(I) Up dominance relation

$$R_A^{U\geq} = \{(y,x) \in U \times U \mid \forall a \in A, \max f(y,a) \geq \max f(x,a)\}, \quad (3.1)$$

if $(y,x) \in R_A^{U\geq}$, we say y is at least up good as x with respect to A;

(II) Down dominance relation

$$R_A^{D\geq} = \{(y,x) \in U \times U \mid \forall a \in A, \min f(y,a) \geq \min f(x,a)\}, \quad (3.2)$$

if $(y,x) \in R_A^{D\geq}$, we say y is at least down good as x with respect to A;

(III) Up-down dominance relation

$$R_A^{UD\geq} = \{(y,x) \in U \times U \mid \forall a \in A, \max f(y,a) \geq \min f(x,a)\}, \quad (3.3)$$

if $(y,x) \in R_A^{UD\geq}$, we say y is at least possible good as x with respect to A;

(IV) Down-up dominance relation

$$R_A^{DU\geq} = \{(y,x) \in U \times U \mid \forall a \in A, \min f(y,a) \geq \max f(x,a)\}, \quad (3.4)$$

if $(y,x) \in R_A^{DU\geq}$, we say y is at least definite good as x with respect to A.

By the definitions of these dominance relations, it can be observed that if a pair of objects (y,x) from $U \times U$ lies in $R_A^{U\geq}$ (or $R_A^{D\geq}$, $R_A^{UD\geq}$, $R_A^{DU\geq}$), then they are perceived as y dominates x; in other words, y may have a better property than x with respect to A in reality.

From the definitions of $R_A^{U\geq}$, $R_A^{D\geq}$, $R_A^{UD\geq}$ and $R_A^{DU\geq}$, the following properties can be easily obtained.

Theorem 3.1. *Let $S = (U, AT, V, f)$ be a disjunctive set-valued information system and $A \subseteq AT$, then*

(1) $R_A^{U\geq} = \bigcap_{a\in A} R_a^{U\geq}$,
(2) $R_A^{D\geq} = \bigcap_{a\in A} R_a^{D\geq}$,
(3) $R_A^{UD\geq} = \bigcap_{a\in A} R_a^{UD\geq}$,
(4) $R_A^{DU\geq} = \bigcap_{a\in A} R_a^{DU\geq}$.

Proof. They are straightforward. □

Theorem 3.2. *Let $R_A^{U\geq}$, $R_A^{D\geq}$, $R_A^{UD\geq}$ and $R_A^{DU\geq}$ be dominance relations in a disjunctive set-valued information system, then*

(1) $R_A^{U\geq}$ *is reflexive, unsymmetric and transitive*;
(2) $R_A^{D\geq}$ *is reflexive, unsymmetric and transitive*;
(3) $R_A^{UD\geq}$ *is reflexive, unsymmetric and intransitive*;
(4) $R_A^{DU\geq}$ *is inreflexive, unsymmetric and transitive.*

Proof. They can be proved from the definitions of these dominance relations. □

Furthermore, we denote by

$$[x]_A^{U\geq} = \{y \in U \mid (y, x) \in R_A^{U\geq}\}, \quad [x]_A^{U\leq} = \{y \in U \mid (x, y) \in R_A^{U\geq}\},$$

where $[x]_A^{U\geq}$ describes objects that up dominate x and $[x]_A^{U\leq}$ describes objects that are up dominated by x in terms of A in a disjunctive set-valued information system; denote by

$$[x]_A^{D\geq} = \{y \in U \mid (y, x) \in R_A^{D\geq}\}, \quad [x]_A^{D\leq} = \{y \in U \mid (x, y) \in R_A^{D\geq}\},$$

where $[x]_A^{D\geq}$ describes objects that down dominate x and $[x]_A^{D\leq}$ describes objects that are down dominated by x in terms of A in a disjunctive set-valued information system; denote by

$$[x]_A^{UD\geq} = \{y \in U \mid (y, x) \in R_A^{UD\geq}\}, \quad [x]_A^{UD\leq} = \{y \in U \mid (x, y) \in R_A^{UD\geq}\},$$

where $[x]_A^{UD\geq}$ describes objects that may dominate x and $[x]_A^{UD\leq}$ describes objects that may be dominated by x in terms of A in a disjunctive set-valued information system, and denote by

$$[x]_A^{DU\geq} = \{y \in U \mid (y, x) \in R_A^{DU\geq}\}, \quad [x]_A^{DU\leq} = \{y \in U \mid (x, y) \in R_A^{DU\geq}\},$$

where $[x]_A^{DU\geq}$ describes objects that must dominate x and $[x]_A^{DU\leq}$ describes objects that must be dominated by x in terms of A in a disjunctive set-valued information system. Obviously, if $S = (U, AT, V, f)$ is a single-valued information system, then the four dominance relations are all degenerated into the dominance relation $R_A^{\geq}$.

From the denotations above, we can conclude the following properties.

Theorem 3.3. *Let $S = (U, AT, V, f)$ be a disjunctive set-valued ordered information system and $A, B \subseteq AT$, then*

(1) *if $B \subseteq A \subseteq AT$, then $R_B^{U\geq} \supseteq R_A^{U\geq} \supseteq R_{AT}^{U\geq}$, $R_B^{D\geq} \supseteq R_A^{D\geq} \supseteq R_{AT}^{D\geq}$, $R_B^{UD\geq} \supseteq R_A^{UD\geq} \supseteq R_{AT}^{UD\geq}$ and $R_B^{DU\geq} \supseteq R_A^{DU\geq} \supseteq R_{AT}^{DU\geq}$;*
(2) *if $B \subseteq A \subseteq AT$, then $[x]_B^{U\geq} \supseteq [x]_A^{U\geq} \supseteq [x]_{AT}^{U\geq}$, $[x]_B^{D\geq} \supseteq [x]_A^{D\geq} \supseteq [x]_{AT}^{D\geq}$, $[x]_B^{UD\geq} \supseteq [x]_A^{UD\geq} \supseteq [x]_{AT}^{UD\geq}$ and $[x]_B^{DU\geq} \supseteq [x]_A^{DU\geq} \supseteq [x]_{AT}^{DU\geq}$;*
(3) $[x_i]_A^{U\geq} = [x_j]_A^{U\geq}$ *iff* $\max f(x_i, a) = \max f(x_j, a)$ $(\forall a \in A)$;
(4) $[x_i]_A^{D\geq} = [x_j]_A^{D\geq}$ *iff* $\min f(x_i, a) = \min f(x_j, a)$ $(\forall a \in A)$;
(5) *if* $\min f(x_i, a) = \min f(x_j, a)$ $(\forall a \in A)$, *then* $[x_i]_A^{UD\geq} = [x_j]_A^{UD\geq}$;
(6) *if* $\max f(x_i, a) = \max f(x_j, a)$ $(\forall a \in A)$, *then* $[x_i]_A^{DU\geq} = [x_j]_A^{DU\geq}$.

Proof. Let $B \subseteq A \subseteq AT$, (1) and (2) are straightforward.

(3) "$\Rightarrow$" If $[x_i]_A^{U\geq} = [x_j]_A^{U\geq}$, it follows from the reflexivity of the dominance relation $R_A^{U\geq}$ that $x_i \in [x_j]_A^{U\geq}$ and $x_j \in [x_i]_A^{U\geq}$, that is $\max f(x_i, a) \geq \max f(x_j, a)$ and $\max f(x_j, a) \geq \max f(x_i, a)$, $(\forall a \in A)$. Hence, we have $\max f(x_i, a) = \max f(x_j, a)$ $(\forall a \in A)$. "$\Leftarrow$" Let $\max f(x_i, a) = \max f(x_j, a)$ $(\forall a \in A)$. If there exists $x \in U$ such that $\max f(x,a) \geq \max f(x_i, a)$ $(\forall a \in A)$, then we get $\max f(x,a) \geq \max f(x_j, a)$ $(\forall a \in A)$. That is to say, if there exists $x \in U$ such that $x \in [x_i]_A^{U\geq}$, then x must belong to $[x_j]_A^{U\geq}$. Thus, $[x_i]_A^{U\geq} = [x_j]_A^{U\geq}$.

(4) "$\Rightarrow$" If $[x_i]_A^{D\geq} = [x_j]_A^{D\geq}$, it follows from the reflexivity of the dominance relation $R_A^{D\geq}$ that $x_i \in [x_j]_A^{D\geq}$ and $x_j \in [x_i]_A^{D\geq}$, that is $\min f(x_i, a) \geq \min f(x_j, a)$ and $\min f(x_j, a) \geq \min f(x_i, a)$, $(\forall a \in A)$. Hence, we have $\min f(x_i, a) = \min f(x_j, a)$ $(\forall a \in A)$. "$\Leftarrow$" Suppose that $\min f(x_i, a) = \min f(x_j, a)$ $(\forall a \in A)$. If there exists $x \in U$ such that $\min f(x,a) \geq \min f(x_i, a)$ $(\forall a \in A)$, then we get $\min f(x,a) \geq \min f(x_j, a)$ $(\forall a \in A)$. That is to say, if there exists $x \in U$ such that $x \in [x_i]_A^{D\geq}$, then x must belong to $[x_j]_A^{D\geq}$. Thus, $[x_i]_A^{D\geq} = [x_j]_A^{D\geq}$.

(5) For $\forall x \in U$, if $x \in [x_i]_A^{UD\geq}$, it follows from the definition of the dominance relation $R_A^{UD\geq}$ that $\max f(x, a) \geq \min f(x_i, a)$ $(\forall a \in A)$. Since the assumption $\min f(x_i, a) = \min f(x_j, a)$ $(\forall a \in A)$, thus $\max f(x, a) \geq \min f(x_j, a)$ $(\forall a \in A)$, i.e., $x \in [x_j]_A^{UD\geq}$. Hence, $[x_i]_A^{UD\geq} \subseteq [x_j]_A^{UD\geq}$ holds. Analogously, we can prove $[x_j]_A^{UD\geq} \subseteq [x_i]_A^{UD\geq}$. Therefore, we have $[x_i]_A^{UD\geq} = [x_j]_A^{UD\geq}$.

(6) For $\forall x \in U$, if $x \in [x_i]_A^{DU\geq}$, it follows from the definition of the dominance relation $R_A^{DU\geq}$ that $\min f(x, a) \geq \max f(x_i, a)$ $(\forall a \in A)$. Since the assumption $\min f(x_i, a) = \min f(x_j, a)$ $(\forall a \in A)$, thus $\min f(x, a) \geq \max f(x_j, a)$ $(\forall a \in A)$, i.e. $x \in [x_j]_A^{DU\geq}$. Hence, $[x_i]_A^{DU\geq} \subseteq [x_j]_A^{DU\geq}$ holds. Analogously, we can prove $[x_j]_A^{DU\geq} \subseteq [x_i]_A^{DU\geq}$. Therefore, we have $[x_i]_A^{DU\geq} = [x_j]_A^{DU\geq}$.

This completes the proof. □

However, the reverse relationships of (5) and (6) in Theorem 3.3 cannot be established in general. For example, let $c(x_1) = \{1, 2\}$, $c(x_2) = \{0, 1\}$ and $c(x_3) = \{0, 2\}$, we have that $[x_1]_A^{UD\geq} = [x_2]_A^{UD\geq} = [x_3]_A^{UD\geq} = \{x_1, x_2, x_3\}$ and $\min c(x_1) = 1 \neq 0 = \min c(x_2) = \min c(x_3)$. And, for instance, let $c(x_1) = \{2, 3\}$, $c(x_2) = \{0, 1\}$ and $c(x_3) = \{0, 2\}$, we obtain that $[x_2]_A^{DU\geq} = [x_3]_A^{DU\geq} = \{x_1\}$, but $\max c(x_2) = 1 \neq 2 = \max c(x_3)$.

Let $U/R_A^{U\geq}$, $U/R_A^{D\geq}$, $U/R_A^{UD\geq}$ and $U/R_A^{DU\geq}$ denote classifications induced by the dominance relations $R_A^{U\geq}$, $R_A^{D\geq}$, $R_A^{UD\geq}$ and $R_A^{DU\geq}$, respectively, which are the family sets $\mathcal{F}^U = \{[x]_A^{U\geq} \mid x \in U\}$, $\mathcal{F}^D = \{[x]_A^{U\geq} \mid x \in U\}$, $\mathcal{F}^{UD} = \{[x]_A^{UD\geq} \mid x \in U\}$ and $\mathcal{F}^{DU} = \{[x]_A^{DU\geq} \mid x \in U\}$. Any element from them will be called a dominance class. All the dominance classes in $U/R_A^{U\geq}$ (or $U/R_A^{D\geq}$, $U/R_A^{UD\geq}$, $U/R_A^{DU\geq}$) do not constitute a partition of U in general. In fact, $\mathcal{F}^U$, $\mathcal{F}^D$ and $\mathcal{F}^{UD}$ all induce a covering of U, i.e. $\bigcup_{x\in U}[x]_A^{U\geq} = U$, $\bigcup_{x\in U}[x]_A^{D\geq} = U$ and $\bigcup_{x\in U}[x]_A^{UD\geq} = U$. However, $\mathcal{F}^{DU}$ can not induce a covering of U because of its inreflexivity in general.

As a depiction of the relationship among the four types of dominance relations in a disjunctive set-valued ordered information system, the following properties can be concluded.

Theorem 3.4. *Let $S = (U, AT, V, f)$ be a disjunctive set-valued ordered information system, $A \subseteq AT$. Then the following implications between the four dominance relations hold*

(1) (I) *implies* (III) : $[(x, y) \in R_A^{U\geq}] \Rightarrow [(x, y) \in R_A^{UD\geq}]$;
(2) (II) *implies* (III) : $[(x, y) \in R_A^{D\geq}] \Rightarrow [(x, y) \in R_A^{UD\geq}]$;
(3) (IV) *implies* (I) : $[(x, y) \in R_A^{DU\geq}] \Rightarrow [(x, y) \in R_A^{U\geq}]$;
(4) (IV) *implies* (II) : $[(x, y) \in R_A^{DU\geq}] \Rightarrow [(x, y) \in R_A^{D\geq}]$;
(5) (IV) *implies* (III) : $[(x, y) \in R_A^{DU\geq}] \Rightarrow [(x, y) \in R_A^{UD\geq}]$.

Proof. They can be proved according to the definitions of the four dominance relations. □

From Theorem 3.3, we can easily obtain the following corollary.

Corollary 3.1. *Let $S = (U, AT, V, f)$ be a disjunctive set-valued ordered information system, $A \subseteq AT$. Then*

(1) $[x]_A^{DU\geq} \subseteq [x]_A^{U\geq} \subseteq [x]_A^{UD\geq}$, $\forall x \in U$;
(2) $[x]_A^{DU\geq} \subseteq [x]_A^{D\geq} \subseteq [x]_A^{UD\geq}$, $\forall x \in U$.

In the following, an illustrative example is employed to understand the four dominance relations.

Example 3.2. Compute the classifications induced by the four dominance relations in Table 2.

From Table 2, we have that

(1) $U/R_{AT}^{U\geqq} = \{[x_1]_{AT}^{U\geqq}, [x_2]_{AT}^{U\geqq}, \ldots, [x_{10}]_{AT}^{U\geqq}\}$,
where

$$[x_1]_{AT}^{U\geqq} = \{x_1\}, \quad [x_2]_{AT}^{U\geqq} = \{x_2\}, \quad [x_3]_{AT}^{U\geqq} = \{x_3, x_7\}, \quad [x_4]_{AT}^{U\geqq} = \{x_4, x_7, x_9, x_{10}\},$$
$$[x_5]_{AT}^{U\geqq} = [x_6]_{AT}^{U\geqq} = \{x_5, x_6\}, \quad [x_7]_{AT}^{U\geqq} = \{x_7\}, \quad [x_8]_{AT}^{U\geqq} = \{x_7, x_8\},$$
$$[x_9]_{AT}^{U\geqq} = [x_{10}]_{AT}^{U\geqq} = \{x_9, x_{10}\};$$

(2) $U/R_{AT}^{D\geqq} = \{[x_1]_{AT}^{D\geqq}, [x_2]_{AT}^{D\geqq}, \ldots, [x_{10}]_{AT}^{D\geqq}\}$,
where

$$[x_1]_{AT}^{D\geqq} = \{x_1, x_7, x_9\}, \quad [x_2]_{AT}^{D\geqq} = \{x_2, x_8\}, \quad [x_3]_{AT}^{D\geqq} = \{x_2, x_3, x_4, x_8, x_{10}\},$$
$$[x_4]_{AT}^{D\geqq} = \{x_4\}, \quad [x_5]_{AT}^{D\geqq} = \{x_5\}, \quad [x_6]_{AT}^{D\geqq} = \{x_5, x_6, x_{10}\},$$
$$[x_7]_{AT}^{D\geqq} = \{x_1, x_7\}, \quad [x_8]_{AT}^{D\geqq} = \{x_2, x_8\}, \quad [x_9]_{AT}^{D\geqq} = \{x_1, x_9\}, \quad [x_{10}]_{AT}^{D\geqq} = \{x_{10}\};$$

(3) $U/R_{AT}^{UD\geq} = \{[x_1]_{AT}^{UD\geq}, [x_2]_{AT}^{UD\geq}, \ldots, [x_{10}]_{AT}^{UD\geq}\}$,
where

$$[x_1]_{AT}^{UD\geq} = [x_7]_{AT}^{UD\geq} = [x_9]_{AT}^{UD\geq} = \{x_1, x_7, x_9, x_{10}\},$$
$$[x_2]_{AT}^{UD\geq} = [x_8]_{AT}^{UD\geq} = \{x_2, x_3, x_7, x_8\}, \quad [x_{10}]_{AT}^{UD\geq} = \{x_9, x_{10}\},$$
$$[x_3]_{AT}^{UD\geq} = \{x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\},$$
$$[x_4]_{AT}^{UD\geq} = \{x_3, x_4, x_7, x_8, x_9, x_{10}\},$$
$$[x_5]_{AT}^{UD\geq} = \{x_5, x_6\}, \quad [x_6]_{AT}^{UD\geq} = \{x_1, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\};$$

(4) $U/R_{AT}^{DU\geq} = \{[x_1]_{AT}^{DU\geq}, [x_2]_{AT}^{DU\geq}, \ldots, [x_{10}]_{AT}^{DU\geq}\}$, where $[x_i]_{AT}^{DU\geq} = \emptyset$, $i \leq 10$.

Let $A = \{a_5\}$, we can get that $U/R_{A}^{DU\geq} = \{[x_1]_{A}^{DU\geq}, [x_2]_{A}^{DU\geq}, \ldots, [x_{10}]_{A}^{DU\geq}\}$, where

$$[x_1]_{A}^{DU\geq} = [x_4]_{A}^{DU\geq} = [x_7]_{A}^{DU\geq} = [x_9]_{A}^{DU\geq} = [x_{10}]_{A}^{DU\geq} = \{x_1, x_7, x_9, x_{10}\},$$
$$[x_2]_{A}^{UD\geq} = [x_3]_{A}^{DU\geq} = \{x_i, i \leq 10\},$$
$$[x_5]_{A}^{DU\geq} = [x_6]_{A}^{D\geq} = [x_8]_{A}^{UD\geq} = \{x_1, x_5, x_6, x_7, x_9, x_{10}\}.$$

From Example 2.3, one can easily notice that

$$[x]_{AT}^{DU\geq} \subseteq [x]_{AT}^{U\geqq} \subseteq [x]_{AT}^{UD\geq},$$
$$[x]_{AT}^{DU\geq} \subseteq [x]_{AT}^{D\geqq} \subseteq [x]_{AT}^{UD\geq}, \quad \forall x \in U.$$

Based on the above analysis, the relationship among the four types of dominance relations in a disjunctive set-valued ordered information system can be summarized, and the corresponding superset-subset relationship graph is depicted in Fig. 1.

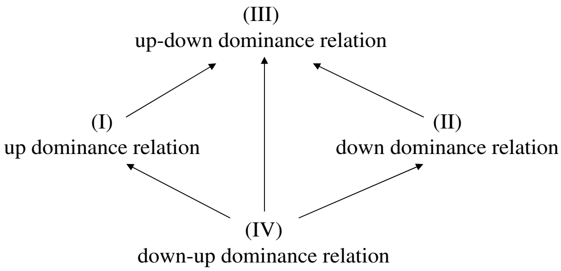

Fig. 1. Superset-subset relationship among the four types of dominance relations.

In Fig. 1, an arrow stands for an implication between two dominance relations. For example, "(I) up dominance relation $\rightarrow$ (III) up-down dominance relation" means (I) $\Rightarrow$ (III). Thus, in a disjunctive set-valued ordered information system, the down-up dominance relation $R_A^{DU\geq}$ is the strongest, while the up-down dominance $R_A^{UD\geq}$ is the weakest.

4. Rough Sets Approaches to Disjunctive Set-Valued Ordered Information Systems

In the section, we investigate the problem of set approximation with respect to the four dominance relations proposed in disjunctive set-valued ordered information systems.

Definition 4.1. Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS. For any $X \subseteq U$ and $A \subseteq AT$, the lower and upper approximations of X with respect to the dominance relation $R_A^{\Delta\geq}$ ($\Delta = U, D, UD, DU$) are defined as follows

$$\underline{R_A^{\Delta\geq}}(X) = \{x \in U \mid (x \cup [x]_A^{\Delta\geq}) \subseteq X\},$$

$$\overline{R_A^{\Delta\geq}}(X) = \{x \in U \mid (x \cup [x]_A^{\Delta\geq}) \cap X \neq \emptyset\}.$$

From Definition 4.1, one can easily notice that $\underline{R_A^{\Delta\geq}}(X)$ is a set of objects that belong to X with certainty, whereas $\overline{R_A^{\Delta\geq}}(X)$ is a set of objects that possibly belong to X. $Bn_A^{\Delta}(X) = \overline{R_A^{\Delta\geq}}(X) - \underline{R_A^{\Delta\geq}}(X)$ denotes the boundary of the rough set.

From Definition 4.1, one can easily obtain the following properties.

Theorem 4.1. *Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS, $X \subseteq U$, $A \subseteq AT$, and $R_A^{\Delta\geq}$ ($\Delta = U, D, UD, DU$) a dominance relation, then*

(1) $\underline{R_A^{\Delta\geq}}(\emptyset) = \overline{R_A^{\Delta\geq}}(\emptyset) = \emptyset$, $\underline{R_A^{\Delta\geq}}(U) = \overline{R_A^{\Delta\geq}}(U) = U$;

(2) $\underline{R_A^{\Delta\geq}}(X) \subseteq X \subseteq \overline{R_A^{\Delta\geq}}(X)$;

(3) $\underline{R_A^{\Delta\geq}}(\underline{R_A^{\Delta\geq}}(X)) = \underline{R_A^{\Delta\geq}}(X)$, $\overline{R_A^{\Delta\geq}}(\overline{R_A^{\Delta\geq}}(X)) = \overline{R_A^{\Delta\geq}}(X)$;

(4) $\underline{R_A^{\Delta\geq}}(X) = \sim \overline{R_A^{\Delta\geq}}(\sim X)$, $\overline{R_A^{\Delta\geq}}(X) = \sim \underline{R_A^{\Delta\geq}}(\sim X)$;

(5) $\underline{R_A^{\Delta\geq}}(X) \subseteq \underline{R_{AT}^{\Delta\geq}}(X)$, $\overline{R_A^{\Delta\geq}}(X) \supseteq \overline{R_{AT}^{\Delta\geq}}(X)$, $Bn_{AT}^{\Delta}(X) \subseteq Bn_A^{\Delta}(X)$.

Theorem 4.2. *Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS, $X, Y \subseteq U$, $A \subseteq AT$, and $R_A^{\Delta\geq}$ ($\Delta = U, D, UD, DU$) a dominance relation, then*

(1) *if $X \subseteq Y$, then $\underline{R_A^{\Delta\geq}}(X) \subseteq \underline{R_A^{\Delta\geq}}(Y)$, $\overline{R_A^{\Delta\geq}}(X) \subseteq \overline{R_A^{\Delta\geq}}(Y)$;*

(2) $\underline{R_A^{\Delta\geq}}(X \cap Y) = \underline{R_A^{\Delta\geq}}(X) \cap \underline{R_A^{\Delta\geq}}(Y)$;

(3) $\overline{R_A^{\Delta\geq}}(X \cup Y) = \overline{R_A^{\Delta\geq}}(X) \cup \overline{R_A^{\Delta\geq}}(Y)$;

(4) $\overline{R_A^{\Delta\geq}}(X \cap Y) \subseteq \overline{R_A^{\Delta\geq}}(X) \cap \overline{R_A^{\Delta\geq}}(Y)$;

(5) $\underline{R_A^{\Delta\geq}}(X \cup Y) \supseteq \underline{R_A^{\Delta\geq}}(X) \cup \underline{R_A^{\Delta\geq}}(Y)$.

The lower and upper approximations of X with respect to the dominance relation $R_A^{\Delta\geq}$ ($\Delta = U, D, UD, DU$) can be used to extract decision rules by a decision maker, where $\underline{R_A^{\Delta\geq}}(X)$ can extract decision rules with certainty, while $Bn_A^{\Delta}(X) = \overline{R_A^{\Delta\geq}}(X) - \underline{R_A^{\Delta\geq}}(X)$ can extract possible decision rules.

The following theorem will establish the relationship among the four types of rough sets.

Theorem 4.3. *Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS, $X \subseteq U$, $A \subseteq AT$, and $R_A^{\Delta\geq}$ ($\Delta = U, D, UD, DU$) a dominance relation, then*

(1) $[y \in \underline{R_A^{UD\geq}}(X)] \Rightarrow [y \in \underline{R_A^{U\geq}}(X)]$, $[y \in \overline{R_A^{U\geq}}(X)] \Rightarrow [y \in \overline{R_A^{UD\geq}}(X)]$;

(2) $[y \in \underline{R_A^{UD\geq}}(X)] \Rightarrow [y \in \underline{R_A^{D\geq}}(X)]$, $[y \in \overline{R_A^{D\geq}}(X)] \Rightarrow [y \in \overline{R_A^{UD\geq}}(X)]$;

(3) $[y \in \underline{R_A^{U\geq}}(X)] \Rightarrow [y \in \underline{R_A^{DU\geq}}(X)]$, $[y \in \overline{R_A^{DU\geq}}(X)] \Rightarrow [y \in \overline{R_A^{U\geq}}(X)]$;

(4) $[y \in \underline{R_A^{D\geq}}(X)] \Rightarrow [y \in \underline{R_A^{DU\geq}}(X)]$, $[y \in \overline{R_A^{DU\geq}}(X)] \Rightarrow [y \in \overline{R_A^{D\geq}}(X)]$;

(5) $[y \in \underline{R_A^{UD\geq}}(X)] \Rightarrow [y \in \underline{R_A^{DU\geq}}(X)]$, $[y \in \overline{R_A^{DU\geq}}(X)] \Rightarrow [y \in \overline{R_A^{UD\geq}}(X)]$.

Proof. (1) For $\forall y \in U$, if $y \in \underline{R_A^{UD\geq}}(X)$, then $[y]_A^{UD\geq} \subseteq X$. From Corollary 3.1, we know $[y]_A^{U\geq} \subseteq [y]_A^{UD\geq}$, hence $[y]_A^{U\geq} \subseteq X$. That is, $y \in \underline{R_A^{U\geq}}(X)$. In addition, if $y \in \overline{R_A^{U\geq}}(X)$, then we have $[y]_A^{U\geq} \cap X \neq \emptyset$. Since $[y]_A^{U\geq} \subseteq [y]_A^{UD\geq}$, we have $[y]_A^{UD\geq} \cap X \neq \emptyset$. Thus, we have $y \in \overline{R_A^{UD\geq}}(X)$.

The proofs of (2), (3), (4) and (5) are all similar to that of (1) and are omitted here. □

From Theorem 4.3, we can easily obtain the following corollary, which gives a depiction of the inclusion relationship among the four types of lower/upper approximations in a disjunctive set-valued ordered information system.

Corollary 4.1. *Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS, $X \subseteq U$, $A \subseteq AT$, and $R_A^{\Delta\geq}$ $(\Delta = U, D, UD, DU)$ a dominance relation, then*

(1) $\underline{R_A^{UD\geq}}(X) \subseteq \underline{R_A^{U\geq}}(X) \subseteq \underline{R_A^{DU\geq}}(X)$;

(2) $\overline{R_A^{UD\geq}}(X) \supseteq \overline{R_A^{U\geq}}(X) \supseteq \overline{R_A^{DU\geq}}(X)$;

(3) $\underline{R_A^{UD\geq}}(X) \subseteq \underline{R_A^{D\geq}}(X) \subseteq \underline{R_A^{DU\geq}}(X)$;

(4) $\overline{R_A^{UD\geq}}(X) \supseteq \overline{R_A^{D\geq}}(X) \supseteq \overline{R_A^{DU\geq}}(X)$.

Uncertainty of a rough set is due to the existence of a borderline region. The greater the borderline region of a rough set, the lower is the accuracy of the rough set. In order to measure the imprecision of a rough set induced by a dominance relation in a disjunctive set-valued ordered information system, we introduce the notion of accuracy measure as follows.

Definition 4.2. Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS, $X \subseteq U$ and $A \subseteq AT$. The accuracy measure of X with respect to the dominance relation $R_A^{\Delta\geq}$ $(\Delta = U, D, UD, DU)$ can be defined as

$$\alpha(R_A^{\Delta\geq}, X) = \frac{|\underline{R_A^{\Delta\geq}}(X)|}{|\overline{R_A^{\Delta\geq}}(X)|} = \frac{|\underline{R_A^{\Delta\geq}}(X)|}{|U| - |\underline{R_A^{\Delta\geq}}(\sim X)|} \tag{4.1}$$

The accuracy measure expresses the degree of completeness of our knowledge about X, given the granularity of $U/R_A^{\Delta\geq}$ $(\Delta = U, D, UD, DU)$. This measure not only depends on the lower approximation of X, but also depends on the lower approximation of $\sim X$ as well.

Theorem 4.4. *Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS, $X \subseteq U$, $A \subseteq AT$, and $R_A^{\Delta\geq}$ $(\Delta = U, D, UD, DU)$ a dominance relation, then*

$$\alpha(R_A^{UD\geq}, X) \leq \alpha(R_A^{U\geq}, X) \leq \alpha(R_A^{DU\geq}, X),$$
$$\alpha(R_A^{UD\geq}, X) \leq \alpha(R_A^{D\geq}, X) \leq \alpha(R_A^{DU\geq}, X).$$

Proof. From Theorem 4.3, they can be easily proved. □

Theorem 4.4 shows that for any set $X \in U$, the accuracy measure of X with respect to the dominance relation $R_A^{DU\geq}$ is the biggest, and that of X with respect to the dominance relation $R_A^{UD\geq}$ is the smallest.

Example 4.1. Continue from Example 3.1. Let $X = \{x_1, x_4, x_7, x_9\}$, $A = \{a_4, a_5\}$, compute the lower/approximations of X with respect to A by using the four types of dominance relations.

By computing, we have that

$$\underline{R_A^{U\geq}}(X) = \{x_1\}, \quad \overline{R_A^{U\geq}}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\},$$

$$\underline{R_A^{D\geq}}(X) = \{x_1, x_4, x_7, x_9\}, \quad \overline{R_A^{D\geq}}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\},$$

$$\underline{R_A^{UD\geq}}(X) = \emptyset, \quad \overline{R_A^{UD\geq}}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\},$$

$$\underline{R_A^{DU\geq}}(X) = \{x_1, x_4, x_7, x_9\}, \quad \overline{R_A^{DU\geq}}(X) = \{x_1, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}.$$

It is obvious that

$$\underline{R_A^{UD\geq}}(X) \subseteq \underline{R_A^{U\geq}}(X) \subseteq \underline{R_A^{DU\geq}}(X), \quad \overline{R_A^{UD\geq}}(X) \supseteq \overline{R_A^{U\geq}}(X) \supseteq \overline{R_A^{DU\geq}}(X);$$

$$\underline{R_A^{UD\geq}}(X) \subseteq \underline{R_A^{D\geq}}(X) \subseteq \underline{R_A^{DU\geq}}(X), \quad \overline{R_A^{UD\geq}}(X) \supseteq \overline{R_A^{D\geq}}(X) \supseteq \overline{R_A^{DU\geq}}(X).$$

By computing, their accuracy measure are as follows

$$\alpha(R_A^{U\geq}, X) = \frac{1}{10}, \quad \alpha(R_A^{D\geq}, X) = \frac{4}{10}, \quad \alpha(R_A^{UD\geq}, X) = 0, \quad \alpha(R_A^{DU\geq}, X) = \frac{4}{9}.$$

Therefore, we have that

$$\alpha(R_A^{UD\geq}, X) \leq \alpha(R_A^{U\geq}, X) \leq \alpha(R_A^{DU\geq}, X),$$
$$\alpha(R_A^{UD\geq}, X) \leq \alpha(R_A^{D\geq}, X) \leq \alpha(R_A^{DU\geq}, X).$$

Based on the above analysis, the inclusion relationships among the four types of lower/upper approximations induced by the dominance relations $R_A^{U\geq}$, $R_A^{D\geq}$, $R_A^{UD\geq}$ and $R_A^{DU\geq}$ in a disjunctive set-valued ordered information system can be concluded. The corresponding superset-subset relationship graphs are depicted in Figs. 2 and 3, respectively.

In Figs. 2 and 3, Δ lower/upper approximation denotes the lower/upper approximation of a set induced by the dominance relation $R_A^{\Delta\geq}$ ($\Delta = U, D, UD, DU$), and an arrow stands for an inclusion between two lower/upper approximation of the set. For example, "(III) UD lower approximation $\rightarrow$ (I) U lower approximation" means (I) $\subseteq$ (III). Thus, in a disjunctive set-valued ordered information system, for a set X, UD lower approximation is the smallest and DU lower approximation is the

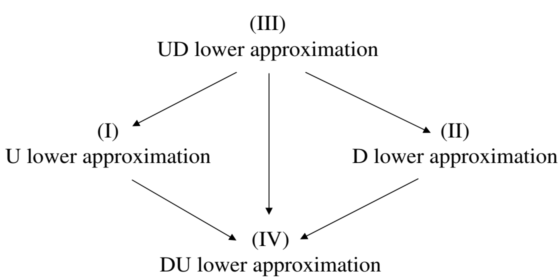

Fig. 2. Inclusion relationship among the four types of lower approximations.

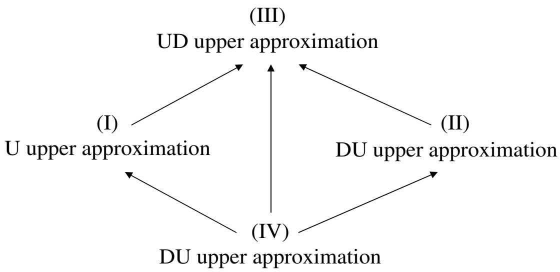

Fig. 3. Inclusion relationship among the four types of upper approximations.

biggest in the four types of lower approximations, while UD upper approximation is the biggest and DU upper approximation is the smallest in the four types of upper approximations.

5. Sorting in Decision-Making

There are two classes of problems in intelligent decision-making: one is to find satisfactory results through ranking with information aggregation, and the other is to find decision rules through relations. In this section, we only focus on the former, i.e. how to make a decision in a disjunctive set-valued ordered information system.

In the following, we introduce the dominance degree between two objects and the whole dominance degree of a object in order to decide the place of each object in final rank.

Definition 5.1. Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS, $A \subseteq AT$, the *dominance degree* between two objects with respect to the dominance relation $R_A^{\Delta\geq}$ $(\Delta = U, D, UD, DU)$ is defined as

$$D_A^{\Delta}(x_i, x_j) = \frac{| \sim [x_i]_A^{\Delta\geq} \cup [x_j]_A^{\Delta\geq} |}{|U|},$$

where $|\cdot|$ denotes the cardinality of a set, $x_i, x_j \in U$.

From the definition, we can get Theorem 5.1 as follows.

Theorem 5.1. $D_A^{\Delta}(x_i, x_j)$ $(\Delta = U, D, UD, DU)$ *have the following properties*

(1) $\frac{1}{|U|} \leq D_A^{\Delta}(x_i, x_j) \leq 1$ $(\Delta = U, D, UD)$, $0 \leq D_A^{\Delta}(x_i, x_j) \leq 1$ $(\Delta = DU)$;
(2) *if* $(x_j, x_k) \in R_A^{\Delta\geq}$, *then* $D_A^{\Delta}(x_i, x_j) \leq D_A^{\Delta}(x_i, x_k)$ $(\Delta = U, D, DU)$;
(3) *if* $(x_j, x_k) \in R_A^{\Delta\geq}$, *then* $D_A^{\Delta}(x_j, x_i) \geq D_A^{\Delta}(x_k, x_i)$ $(\Delta = U, D, DU)$.

Proof. (1) is straightforward.
(2) When $\Delta = U, D, DU$, it follows from Theorem 3.2 that the dominance relation $R_A^{\Delta\geq}$ is transitive. Hence, if $(x_j, x_k) \in R_A^{\Delta\geq}$, we have that $[x_j]_A^{\Delta\geq} \subseteq [x_k]_A^{\Delta\geq}$.

Therefore

$$\begin{aligned} &D_A^{\Delta}(x_i,x_j) - D_A^{\Delta}(x_i,x_k) \\ &\quad = \frac{1}{|U|}(|\sim [x_i]_A^{\Delta\geq} \cup [x_j]_A^{\Delta\geq}| - |\sim [x_i]_A^{\Delta\geq} \cup [x_k]_A^{\Delta\geq}|) \\ &\quad \leq \frac{1}{|U|}(|\sim [x_i]_A^{\Delta\geq} \cup [x_k]_A^{\Delta\geq}| - |\sim [x_i]_A^{\Delta\geq} \cup [x_k]_A^{\Delta\geq}|) \\ &\quad = 0. \end{aligned}$$

(3) Similar to (2), we have that $[x_j]_A^{\Delta\geq} \subseteq [x_k]_A^{\Delta\geq}$, hence $\sim [x_j]_A^{\Delta\geq} \supseteq \sim [x_k]_A^{\Delta\geq}$. Thus

$$\begin{aligned} &D_A^{\Delta}(x_j,x_i) - D_A^{\Delta}(x_k,x_i) \\ &\quad = \frac{1}{|U|}(|\sim [x_j]_A^{\Delta\geq} \cup [x_i]_A^{\Delta\geq}| - |\sim [x_k]_A^{\Delta\geq} \cup [x_i]_A^{\Delta\geq}|) \\ &\quad \geq \frac{1}{|U|}(|\sim [x_k]_A^{\Delta\geq} \cup [x_i]_A^{\Delta\geq}| - |\sim [x_k]_A^{\Delta\geq} \cup [x_i]_A^{\Delta\geq}|) \\ &\quad = 0. \end{aligned}$$

This completes the proof. □

From Definition 5.1, let $(x_i,x_j) \in U \times U$, we can construct a *dominance relation matrix* with respect to A induced by the dominance relation $R_A^{\Delta\geq}$ ($\Delta = U, D, UD, DU$). From this matrix, the *whole dominance degree* of each object can be calculated according to the following formula

$$D_A^{\Delta}(x_i) = \frac{1}{|U|-1}\sum_{j\neq i} D_A^{\Delta}(x_i,x_j), \quad x_i,x_j \in U.$$

From the whole dominance degree of each object on the universe, we can rank all objects according to the number of $D_A^{\Delta}(x_i)$ $R_A^{\Delta\geq}$ ($\Delta = U, D, UD, DU$), a larger number implies a better object. This idea can be understood by the following example. For simplicity, we only consider the decision induced by the dominance relation $R_{AT}^{UD\geq}$.

Example 5.1. Continue from Example 3.2, rank objects in U according to the dominance relation $R_{AT}^{UD\geq}$.

By computing, we can get the dominance relation matrix as follows

$$D_{AT}^{\Delta} = \begin{pmatrix} 1 & 0.7 & 0.9 & 0.9 & 0.6 & 1 & 1 & 0.7 & 1 & 0.8 \\ 0.7 & 1 & 1 & 0.9 & 0.6 & 0.8 & 0.7 & 1 & 0.7 & 0.6 \\ 0.4 & 0.5 & 1 & 0.7 & 0.3 & 0.8 & 0.4 & 0.5 & 0.4 & 0.3 \\ 0.7 & 0.7 & 1 & 1 & 0.4 & 0.9 & 0.7 & 0.7 & 0.7 & 0.6 \\ 0.8 & 0.8 & 1 & 0.8 & 1 & 1 & 0.8 & 0.8 & 0.8 & 0.8 \\ 0.6 & 0.4 & 0.9 & 0.7 & 0.4 & 1 & 0.6 & 0.4 & 0.6 & 0.4 \\ 1 & 0.7 & 0.9 & 0.9 & 0.6 & 1 & 1 & 0.7 & 1 & 0.8 \\ 0.7 & 1 & 1 & 0.9 & 0.6 & 0.8 & 0.7 & 1 & 0.7 & 0.6 \\ 1 & 0.7 & 0.9 & 0.9 & 0.6 & 1 & 1 & 0.7 & 1 & 0.8 \\ 1 & 0.8 & 1 & 1 & 0.8 & 1 & 1 & 0.8 & 1 & 1 \end{pmatrix}$$

Therefore, we have that

$$D^{\Delta}_{AT}(x_1) = 0.84, \quad D^{\Delta}_{AT}(x_2) = 0.78, \quad D^{\Delta}_{AT}(x_3) = 0.48, \quad D^{\Delta}_{AT}(x_4) = 0.71,$$
$$D^{\Delta}_{AT}(x_5) = 0.84, \quad D^{\Delta}_{AT}(x_6) = 0.56, \quad D^{\Delta}_{AT}(x_7) = 0.84, \quad D^{\Delta}_{AT}(x_8) = 0.78,$$
$$D^{\Delta}_{AT}(x_9) = 0.84, \quad D^{\Delta}_{AT}(x_{10}) = 0.93.$$

In the following, ranking objects according to the number of $D^{\Delta}_{AT}(x_i)$, a object with larger number implies a better object.

$$x_{10} \succcurlyeq \begin{pmatrix} x_1 \\ x_5 \\ x_7 \\ x_9 \end{pmatrix} \succcurlyeq \begin{pmatrix} x_2 \\ x_8 \end{pmatrix} \succcurlyeq x_4 \succcurlyeq x_6 \succcurlyeq x_3.$$

6. Criterion Reduction to Disjunctive Set-Valued OIS

In this section, the approaches to the criterion reductions in a disjunctive set-valued ordered information system are presented by using the discernibility matrices, the superset-subset relationship among the four types of criterion reductions are established, and an illustrative examples is employed to show their mechanisms as well.

Firstly, we give the definitions of criterion reductions of a disjunctive set-valued ordered information system.

Definition 6.1. Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS and $A \subseteq AT$. If $R^{\Delta\geq}_{A} = R^{\Delta\geq}_{AT}$ and $R^{\Delta\geq}_{B} \neq R^{\Delta\geq}_{AT}$ for any $B \subset A$, then we call A a Δ criterion reduction of S.

When Δ equals to U, D, UD and DU, the corresponding criterion reduction can be called *up criterion reduction*, *down criterion reduction*, *up-down criterion reduction* and *down-up criterion reduction*, respectively.

Theorem 6.1. *Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS, $A \subseteq AT$. If A is a Δ ($\Delta =U$, D, UD, DU) criterion reduction, then $D^{\Delta}_{A}(x_i, x_j) = D^{\Delta}_{AT}(x_i, x_j)$, $x_i, x_j \in U$.*

Proof. It can be proved from Definition 5.1 and Definition 6.1. □

It is obvious that a criterion reduction of a disjunctive set-valued OIS is a minimal attribute subset satisfying $R^{\Delta\geq}_{A} = R^{\Delta\geq}_{AT}$. An attribute $a \in AT$ is called dispensable with respect to $R^{\Delta\geq}_{AT}$ if $R^{\Delta\geq}_{AT} = R^{\Delta\geq}_{(AT-\{a\})}$; otherwise a is called indispensable. The set of all indispensable attributes is called the core with respect to the dominance relation $R^{\Delta\geq}_{AT}$ and is denoted by $core^{\Delta}(AT)$. An attribute in the core must be in every criterion reduction (like the case in complete/incomplete OIS, an OIS may have many reductions, denoted by $red^{\Delta}(AT)$). Thus $core^{\Delta}(AT) = \bigcap red^{\Delta}(AT)$. The core may be an empty set.

Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS, $A \subseteq AT$. For convenient representation, let Δ =U, D, UD, DU, denote by

$$Dis^{\Delta}(x,y) = \{a \in A \mid (x,y) \notin R^{\Delta\geq}_{\{a\}}\},$$

then we call $Dis^{\Delta}(x,y)$ the Δ discernibility attribute set between x and y, and

$$\mathbf{Dis}^{\Delta} = (Dis^{\Delta}(x,y) : x, y \in U)$$

the Δ discernibility matrix of disjunctive set-valued OIS. Clearly, for $\forall x, y \in U$ we have $Dis^{\Delta}(x,y) \cap Dis^{\Delta}(y,x) = \emptyset$.

The following property provides a judgement method of a Δ criterion reduction of disjunctive set-valued OIS.

Theorem 6.2. *Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS, $A \subseteq AT$, and $Dis^{\Delta}(x,y)$ the Δ discernibility attributes set of S with respect to $R^{\Delta\geq}_{AT}$, then $R^{\Delta\geq}_{AT} = R^{\Delta\geq}_{A}$ iff $A \cap Dis^{\Delta}(x,y) \neq \emptyset$ (where $Dis^{\Delta}(x,y) \neq \emptyset$).*

Proof. "$\Rightarrow$" Let $R^{\Delta\geq}_{AT} = R^{\Delta\geq}_{A}$, from the definition of the dominance relation, we have that for arbitrary $x \in U$, $[x]^{\Delta\geq}_{AT} = [x]^{\Delta\geq}_{A}$ holds. If some $y \notin [x]^{\Delta\geq}_{AT}$, then $y \notin [x]^{\Delta\geq}_{A}$. Therefore, there exists $a \in A$ such that $(x,y) \notin [x]^{\Delta\geq}_{\{a\}}$. So one has $a \in Dis^{\Delta}(x,y)$. Hence, when $Dis^{\Delta}(x,y) \neq \emptyset$ we have $A \cap Dis^{\Delta}(x,y) \neq \emptyset$.

"$\Leftarrow$" From the definition of the Δ discernibility attribute set, we know that if $(x,y) \notin [x]^{\Delta\geq}_{AT}$ for any $(x,y) \in U \times U$, then $Dis^{\Delta}(x,y) \neq \emptyset$. And since $A \cap Dis^{\Delta}(x,y) \neq \emptyset$, there exists $a \in A$ such that $a \in Dis^{\Delta}(x,y)$, i.e. $(x,y) \notin [x]^{\Delta\geq}_{\{a\}}$. So $(x,y) \notin [x]^{\Delta\geq}_{A}$. Hence $R^{\Delta\geq}_{AT} \supseteq R^{\Delta\geq}_{A}$. On the other hand, it follows from $A \subseteq AT$ that $R^{\Delta\geq}_{AT} \subseteq R^{\Delta\geq}_{A}$. Hence, one has $R^{\Delta\geq}_{AT} = R^{\Delta\geq}_{A}$.

This completes the proof. □

Definition 6.2. Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS, $A \subseteq AT$, and $Dis^{\Delta}(x,y)$ the Δ discernibility attributes set of S with respect to $R^{\Delta\geq}_{AT}$. Denote by

$$M^{\Delta} = \bigwedge \Big\{ \bigvee \{a : a \in Dis^{\Delta}(x,y)\} : x, y \in U \Big\},$$

then M^{Δ} is referred to as the Δ discernibility function.

By using the Δ discernibility function, we can design the approach to the Δ criterion reduction in a disjunctive set-valued OIS as follows.

Theorem 6.3. *Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS. The minimal disjunctive normal form of discernibility function M^{Δ} is*

$$M^{\Delta} = \bigvee_{k=1}^{t} \left(\bigwedge_{s=1}^{q_k} a_{i_s} \right),$$

where $\bigwedge_{s=1}^{q_k} a_{i_s}$ represents the conjunction operation among elements in a_{i_s} and $\bigvee_{k=1}^{t}(\bigwedge_{s=1}^{q_k} a_{i_s})$ denotes the disjunction operation among sets. Denote by

$B_k = \{a_{i_s} : s = 1, 2, \ldots, q_k\}$, then $\{B_k : k = 1, 2, \ldots, t\}$ are the set of all Δ criterion reductions of this system. In fact, these B_k is obtained from M^{Δ} by applying the multiplication and absorption laws, which satisfies that every element in B_k only appears one time.

Proof. *It follows directly from Theorem 6.1 and the definition of minimal disjunctive normal form of the Δ discernibility function.* □

Theorem 6.3 provides a kind of practical approaches to the four criterion reductions in a disjunctive set-valued ordered information system.

In the following, an illustrative example is employed to analyze the mechanism of this kind of approach. For simplicity, we only discuss the up-down criterion reduction.

Example 6.1. Continue from Example 3.1, compute all up-down criterion reductions in Table 2.

By computing, we can obtain the UD discernibility matrix of this system (see Table 3).

Hence, we have that

$$\begin{aligned} M^{\Delta} &= (a_2 \vee a_3) \wedge a_3 \wedge a_1 \wedge (a_4 \vee a_5) \wedge a_4 \wedge (a_1 \vee a_5) \wedge a_5 \wedge (a_1 \vee a_3 \vee a_5) \\ &\quad \wedge (a_1 \vee a_3) \wedge (a_3 \vee a_5) \wedge (a_1 \vee a_4 \wedge a_5) \wedge a_2 \\ &= a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_5. \end{aligned}$$

Therefore, $\{a_1, a_2, a_3, a_4, a_5\}$ is a unique up-down criterion reduction for this system, that is, any criterion cannot be eliminated from Table 2 under the dominance relation $R_{AT}^{UD \geq}$.

In succussion, we reveal the implication relationship among the four types of criterion reductions in disjunctive set-valued order information systems.

Table 3. The UD discernibility matrix of Table 2.

x_i/x_j	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	$\emptyset$	a_2, a_3	a_3	a_3	a_1	$\emptyset$	$\emptyset$	a_2, a_3	$\emptyset$	a_3
x_2	a_4, a_5	$\emptyset$	$\emptyset$	a_4	a_1, a_5	a_5	a_4, a_5	$\emptyset$	a_4, a_5	a_5
x_3	a_1, a_5	$\emptyset$	$\emptyset$	$\emptyset$	a_1, a_5	a_5	a_1, a_5	$\emptyset$	a_1, a_5	a_1, a_3, a_5
x_4	a_1	a_2	$\emptyset$	$\emptyset$	a_1	$\emptyset$	a_1	a_2	a_1	a_1, a_3
x_5	a_4, a_5	a_2	$\emptyset$	a_4	$\emptyset$	$\emptyset$	a_4, a_5	a_2	a_4, a_5	a_3, a_5
x_6	a_4, a_5	a_2	$\emptyset$	a_4	$\emptyset$	$\emptyset$	a_4, a_5	a_2	a_4, a_5	a_3, a_5
x_7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	a_1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	a_3
x_8	a_1, a_4, a_5	$\emptyset$	$\emptyset$	a_4	a_1	$\emptyset$	a_1, a_4, a_5	$\emptyset$	a_1, a_4, a_5	a_1, a_3, a_5
x_9	$\emptyset$	a_2	$\emptyset$	$\emptyset$	a_1	$\emptyset$	$\emptyset$	a_2	$\emptyset$	$\emptyset$
x_{10}	$\emptyset$	a_2	$\emptyset$	$\emptyset$	a_1	$\emptyset$	$\emptyset$	a_2	$\emptyset$	$\emptyset$

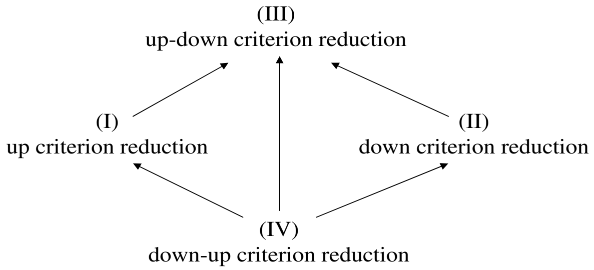

Fig. 4. Superset-subset relationship among the four types of criterion reductions.

Theorem 6.4. *Let $S = (U, AT, V, f)$ be a disjunctive set-valued OIS, $A \subseteq AT$. Then the following implications between properties of respective criterion reductions hold*

(1) $[R_A^{U\geq} = R_{AT}^{U\geq}] \Rightarrow [R_A^{DU\geq} = R_{AT}^{DU\geq}]$,
(2) $[R_A^{D\geq} = R_{AT}^{D\geq}] \Rightarrow [R_A^{DU\geq} = R_{AT}^{DU\geq}]$,
(3) $[R_A^{UD\geq} = R_{AT}^{UD\geq}] \Rightarrow [R_A^{DU\geq} = R_{AT}^{DU\geq}]$,
(4) $[R_A^{UD\geq} = R_{AT}^{UD\geq}] \Rightarrow [R_A^{U\geq} = R_{AT}^{U\geq}]$,
(5) $[R_A^{UD\geq} = R_{AT}^{UD\geq}] \Rightarrow [R_A^{D\geq} = R_{AT}^{D\geq}]$.

Proof. (1) Suppose that $R_A^{DU\geq} \neq R_{AT}^{DU\geq}$ when $R_A^{U\geq} = R_{AT}^{U\geq}$. Since $A \subseteq AT$, we have that $R_{AT}^{DU\geq} \subseteq R_A^{DU\geq}$. Hence, there exists a order pair $(x, y) \in R_A^{DU\geq}$ and $(x, y) \notin R_{AT}^{DU\geq}$. From (3) of Theorem 3.4, we can get that $(x, y) \in R_A^{U\geq}$ and $(x, y) \notin R_{AT}^{U\geq}$, i.e. $R_A^{U\geq} \neq R_{AT}^{U\geq}$. This yields a contradiction. Thus $R_A^{DU\geq} = R_{AT}^{DU\geq}$ holds if $R_A^{U\geq} = R_{AT}^{U\geq}$.

The proofs of (2), (3), (4) and (5) is similar to that of (1). □

Based on the above analysis, the implication relationship among the four types of criterion reductions in a disjunctive set-valued ordered information system can be summarized, and the corresponding superset-subset relationship graph is depicted in Fig. 4.

In Fig. 4, an arrow stands for an implication between two dominance relations. For example, "(I) up criterion reduction → (III) up-down criterion reduction" means (I) ⇒ (III). Thus, in a disjunctive set-valued ordered information system, for each UD criterion reduction A of S, there must exist a U criterion reduction B of S, such that B is a subset of A. The interpretation of the rest implications are all similar to that of (I) ⇒ (III). Note that the four types of criterion reductions can degenerate into the classical criterion reduction of an order information system.

7. An Application for Venture Investment

Venture capital has become an increasingly important source of financing for new companies, particularly when such companies are operating on the frontier of emerging technologies and markets. It plays an essential role in the entrepreneurial process.[42] For an investor or decision maker, he may need to adopt a better one from some possible investment projects or find some directions from existing successful investment projects before investing. The purpose of this section is, through a venture investment issue, to illustrate how to make a decision by using the approaches proposed in this paper.

Let us consider an investment issue of a venture investment company. There are five investment projects x_i $(i = 1, 2, \ldots, 5)$ can be considered. They can be evaluated from the view of venture factors. Venture factors are classified into six factors, which are market venture, technology venture, management venture, environment venture, production venture and finance venture. These six factors are all increasing preference and the value of each project under each factor is given by an evaluation expert through a set value. Table 4 is an evaluation table about venture investment given by an expert, where $U = \{x_1, x_2, x_3, x_4, x_5\}$, $AT = \{$Market, Technology, Management, Environment, Production, Finance$\}$. For convenience, in the sequel, M_1, T, M_2, E, P, F will stand for Market, Technology, Management, Environment, Production and Finance, respectively. For convenience, we only consider Up dominance relation in this case study. Similarly, one can obtain the corresponding decisions through using the rest three dominance relations.

From Table 4, we have that

$$U/R_{AT}^{U\geqq} = \{[x_1]_{AT}^{U\geqq},\ [x_2]_{AT}^{U\geqq},\ [x_3]_{AT}^{U\geqq},\ [x_4]_{AT}^{U\geqq},\ [x_5]_{AT}^{U\geqq}\},$$

where $[x_1]_{AT}^{U\geqq} = \{x_1, x_3\}$, $[x_2]_{AT}^{U\geqq} = \{x_1, x_2, x_3, x_4\}$, $[x_3]_{AT}^{U\geqq} = \{x_3\}$, $[x_4]_{AT}^{U\geqq} = \{x_1, x_3, x_4\}$ and $[x_5]_{AT}^{U\geqq} = \{x_1, x_2, x_3, x_4, x_5\}$.

From the definition of dominance degree, we can get the dominance relation matrix of this table with respect to $U/R_{AT}^{U\geqq}$ as

$$\begin{pmatrix} 1 & 1 & 0.8 & 1 & 1 \\ 0.6 & 1 & 0.4 & 0.8 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0.8 & 1 & 0.6 & 1 & 1 \\ 0.4 & 0.8 & 0.2 & 0.6 & 1 \end{pmatrix}$$

Table 4. An interval ordered information system about venture investment.

Projects	M_1	T	M_2	E	P	F
x_1	$\{3,4\}$	$\{4,5\}$	$\{3,4\}$	$\{3,4\}$	$\{2,3\}$	$\{4,5\}$
x_2	$\{1,2\}$	$\{1,2\}$	$\{1,3\}$	$\{1,3\}$	$\{2,3\}$	$\{1,3\}$
x_3	$\{3,4\}$	$\{4,5\}$	$\{3,5\}$	$\{3,4\}$	$\{3,5\}$	$\{4,5\}$
x_4	$\{2,3\}$	$\{4,5\}$	$\{2,3\}$	$\{2,4\}$	$\{2,3\}$	$\{3,5\}$
x_5	$\{1,2\}$	$\{1,2\}$	$\{1,3\}$	$\{1,2\}$	$\{2,3\}$	$\{1,3\}$

Table 5. The discernibility matrix of Table 4.

x_i/x_j	x_1	x_2	x_3	x_4	x_5
x_1	Ø	M_1TM_2EF	Ø	M_1M_2	M_1TM_2EF
x_2	Ø	Ø	Ø	Ø	E
x_3	M_2P	M_1TM_2EPF	Ø	M_1M_2P	M_1TM_2EPF
x_4	Ø	M_1TEF	Ø	Ø	M_1TEF
x_5	Ø	$\{E\}$	Ø	Ø	Ø

Therefore, one can obtain that

$$D^U_{AT}(x_1) = 0.95, \quad D^U_{AT}(x_2) = 0.70, \quad D^U_{AT}(x_3) = 1.00,$$
$$D^U_{AT}(x_4) = 0.85 \quad \text{and} \quad D^U_{AT}(x_5) = 0.60.$$

As follows, we rank these five projects according to the number of $D^U_{AT}(x_i)$. A project with whole dominance degree implies that it has higher investment venture.

$$x_3 \succcurlyeq x_1 \succcurlyeq x_4 \succcurlyeq x_2 \succcurlyeq x_5.$$

Thus, the investment venture of project x_3 is highest and that of project x_5 is lowest. The decision maker may select the project x_5 to invest.

To extract much simpler criterion representation, we compute criterion reductions of this information system. The criterion reductions of this information system can be obtained by the proposed reduction approach in this paper. Table 5 is the discernibility matrix of this disjunctive set-valued OIS, where values of $Dis^U(x_i, x_j)$ for any pair (x_i, x_j) of projects are placed.

From Table 5, one can obtain that

$$\begin{aligned} M^U &= (M_1TM_2EF) \wedge (M_1M_2) \wedge E \wedge (M_2P) \wedge (M_1TM_2EPF) \\ &\quad \wedge (M_1M_2P) \wedge (M_1TEF) \\ &= (M_1 \vee M_2) \wedge E \wedge (M_2 \vee P) \\ &= (M_2 \wedge E) \vee (M_1 \wedge E \wedge P) \vee (M_1 \wedge E \wedge M_2). \end{aligned}$$

Hence, there are three criterion reductions in this disjunctive set-valued ordered information system about venture investment, which are {Management, Environment}, {Market, Environment, Production} and {Market, Environment, Management}. From this result, we know that the venture factor Environment is indispensable for this decision problem. Therefore, market venture, management venture, environment venture and production venture are four important venture factors for this investment issue.

8. Conclusions

To recapitulate, a set-valued information systems is an important formal framework for the development of decision support systems. Because of the existence of set-values, it can be classified into two categories: disjunctive and conjunctive. We focused on the former in this paper.

According to four types of dominance relations in the disjunctive set-valued information systems, a dominance-based rough sets approach have been established, which is mainly based on substitution of the indiscernibility relation by the dominance relations. A new sorting approach to all objects in a given system have been presented, which is based on the corresponding classes of objects induced by a dominance relation. To simplify a disjunctive set-valued ordered information system, criterion reductions of disjunctive set-valued ordered information systems have been investigated, which eliminate only that information that is not essential from the view of the ordering of objects. It should be noted that the interrelationships among the four types of dominance relations have been established as well, which include corresponding dominance classes, rough sets approaches, sorting for objects and criterion reductions. It is hoped that the present study can further stimulate investigation for decision making in ordered information systems.

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