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Deviation Degree: A Perspective on Score Functions in Hesitant Fuzzy Sets

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Abstract Score functions play an important role in ranking hesitant fuzzy elements (HFEs) and hesitant fuzzy sets (HFSs). Currently, various kinds of HFE and HFS score functions have been investigated in the literature. However, the essential characteristic and generation mechanism of these score functions have not been systematically studied. To address these issues, this paper introduces an axiomatic definition of deviation degree measure and proposes a general form of dual HFE and HFS deviation score functions, from which a family of existing HFE and HFS score functions can be derived. Besides, we develop two ranking methods based on a pair of dual deviation score functions for distinguishing HFEs and HFSs that are indiscernible by a single score function. Moreover, the mathematical and behavioral properties of HFS deviation score functions are analyzed for applying them in practice. Finally, the proposed ranking method for HFSs is applied to the multi-criteria decision-making problems with hesitant fuzzy information.

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1 Introduction

Hesitant fuzzy set (HFS), introduced initially by Torra et al. [1, 2], is an interesting extension of Zadeh's fuzzy set [3]. It relaxes the restriction of memberships in the classical fuzzy set and allows the membership of an element to a set by several possible values [4, 5]. Accordingly, it is suitable for solving the problems in which one is hesitant and irresolute for one thing or another to reach a final agreement. For the distinctive characteristic of expressing the doubtful information, hesitant fuzzy sets have attracted more and more researchers to devote to further promoting the theoretical studies [6–13] and extending the applications in the fields of decision analysis, medical diagnosis, pattern recognition, and so on [14–20].

A growing number of studies focus on the measurements for HFEs and HFSs. It is not only because of their fundamental importance in theoretical studies, but also the indispensability in almost all of the application fields. Xu [5] initially defined the HFE distance, correlation coefficient, similarity, and cross-entropy measure; then he [6] developed a series of hesitant order weighed distance and similarity measures for HFSs. Later, Farhadinia [7–9] investigated the similarity, distance measure, and correlation coefficient for the HFSs and the extend HFSs. The axiomatic definitions and formulas of entropy were first introduced for HFEs and HFSs in [10]. Recently, Wei [21] designed a more general family of concrete entropy formulas, which could measure both fuzziness and hesitation degree of HFEs and perform better than the existing ones.

Chen gave a single measure for HFEs called information energy [22]. Xia [23] defined a commonly used score function which calculates the score value of a hesitant fuzzy element by averaging the membership values in HFEs. These classical measurements have been successfully applied to various real problems.

By analyzing the current measurements for HFEs and HFSs [5–10, 21–23], we can divide them into two categories: the relationship measures (distance, similarity, correlation coefficient, and cross-entropy) [5, 7–10] and the single element measures (entropy, information energy, and score function) [21–23]. The relationship measures are generally used in pattern recognition and clustering analysis [24, 25], and the single element measures are of great value especially in decision-making problems owing to the advantages of depicting the performances of HFEs and HFSs [26, 27]. Score function, as one of the most critical latter measures, is proposed to be a useful tool for converting the HFE (HFS) data into a single numerical value. By describing the magnitudes of HFEs (HFSs), a score function can help to derive the corresponding comparison laws for HFEs (HFSs) and then provide a much more convenient way to rank the involved HFEs (HFSs). Moreover, it provides a much more convenient way to rank the alternatives in the decision-making problems with hesitant fuzzy information (HFMADM).

Since Xia proposed the first score function in [23], researchers have designed diverse kinds of score functions to obtain the appropriate score values for achieving a more reliable ranking. By considering Xia's score function and the variance value of the memberships, Liao [27] designed a two-step algorithmic approach, Score value-Variance, to rank HFEs. Zhang [28] developed a kind of score function which is a generalization of Xia's score function by considering the variance of memberships in HFEs. Farhadinia [29] proposed a novel improved score function with the position weighting information. Zhou [30] gave an order relation between two HFEs based on score values and deviation function. More recently, Farhadinia [31] presented a series of score functions for HFEs and described a scheme for choosing an appropriate score function from a set of score functions. However, most of the above score functions are defined for HFEs, and these measures are usually used to distinguish HFEs, i.e., the ingredients of HFSs. In [31], Farhadinia also investigated score functions for HFSs from HFE score functions, which greatly enriches the score function theory in hesitant fuzzy sets. Based on score values calculated from the HFS score functions, Farhadinia also built a quick method for determining the priorities of the alternatives in a hesitant fuzzy expressed multi-attribute decision-making problem.

Although many excellent contributions about score functions with varied forms have been made in the context of hesitant fuzzy sets, there are still some vital issues to be addressed. Does there exist a general HFE (HFS) score function which can depict the essence of the HFE (HFS) score functions? If it exists, can it degenerate into the current individual HFE (HFS) score functions, and is it with a more effective performance? This study aims to explore the nature of HFE (HFS) score functions from the perspective of deviation degree and investigate the versatility of the newly proposed score functions. Inspired by the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), a new HFE (HFS) deviation score function is first defined to reveal the essence of score functions by computing the deviation degree between an HFE (HFS) and the two extreme HFEs (HFSs). It is also proved that such HFE (HFS) deviation score functions are generalizations of the existing HFE (HFS) score functions in the literature. Moreover, we develop two ranking methods based on a pair of dual deviation score functions for ranking HFEs and HFSs more finely. Besides, we analyze the mathematical and behavioral properties of deviation score functions especially for HFSs and provide useful rules to select the appropriate HFS deviation score functions in practical applications.

This paper is organized as follows. Some basic concepts about hesitant fuzzy sets and score functions are reviewed in Sect. 2. Section 3 presents concepts of deviation degree and deviation score function for HFEs and develops a ranking method based on a pair of dual HFE deviation score values. Section 4 extends the concepts of deviation score function and the ranking method for HFEs to those for HFSs. Section 5 discusses the mathematical and behavioral properties of HFS deviation score functions for actual use. The practicality and effectiveness of the new HFS ranking method is illustrated by a human resource management example in Sect. 6. Section 7 concludes this paper.

2 Preliminaries

This section is devoted to describing the basic notions about hesitant fuzzy sets and reviewing some important score functions in the literature. Throughout this paper, $X = \{x_1, x_2, \ldots, x_n\}$ is used to denote the universe of discourse.

2.1 Basic Concepts of Hesitant Fuzzy Sets

Definition 1 [1, 23] Let X be a discourse set. A hesitant fuzzy set on X is in terms of a function h that when applied to X returns a subset of values in the interval [0, 1].

To be easily understood, Xia and Xu [23] expressed an HFS by

$$E = \{ <x, h_E(x)> | x \in X\}, \tag{1}$$

where $h_E(x)$ is a set of some different values in [0, 1], representing the possible membership degrees of the element x to E. Xia and Xu [23] named $h_E(x)$ a hesitant fuzzy element. They also called $\{ <x, h_E(x) = \{0\}> | x \in X\}$ and $\{ <x, h_E(x) = \{1\}> | x \in X\}$ the empty hesitant set and the full hesitant set, respectively. Similarly, $h(x) = \{0\}$ and $h(x) = \{1\}$ were called the empty hesitant element and the full hesitant element, respectively.

Example 1 Suppose $X = \{x_1, x_2, x_3\}$ is the universe of discourse. An HFS on X is given as $E = \{ <x_1, \{0.7, 0.1\}>, <x_2, \{0.4\}>, <x_3, \{0.6, 0.4, 0.2\}> \}$. Three HFEs $h_E(x_1) = \{0.7, 0.1\}$, $h_E(x_2) = \{0.4\}$ and $h_E(x_3) = \{0.6, 0.4, 0.2\}$ represent the sets of possible memberships for x_1, x_2, and x_3 to E, respectively.

The values in an HFE are usually out of order. Then it is necessary to arrange them in any order for convenience [15]. Suppose the values in h are arranged in decreasing order, and γ_i is the ith largest value in h. Meanwhile, the numbers of membership values for different HFEs may be different. To compare and calculate the relative measurements between two HFEs, Xu et al. [15, 23] suggested that one should extend the shorter HFE by repeating some elements until both of them have the same length. Xu and Zhang also proposed a general method equipped with a parameter η for identifying the extension value.

Definition 2 [15] Let $h = \{\gamma_1, \gamma_2, \ldots, \gamma_l\}$ be an HFE, γ_1 and γ_l be the maximum and minimum values in h, respectively. Then we call $\overline{\gamma} = \eta\gamma_1 + (1-\eta)\gamma_l$ an extension value, where η $(0 \le \eta \le 1)$ is a risk attitude parameter.

The parameter η is used to accommodate decision makers with varying risk preferences. The extreme risk-seeking preference is modeled by $\eta = 1$; the extreme risk-averse preference by $\eta = 0$; and the risk-neutral preference by $\eta = \frac{1}{2}$.

Remark 1 The following assumptions should be kept in mind throughout the paper.

(1) The number of memberships in an HFE is finite, and membership values are in decreasing order.
(2) $\mathbf{0}$: the empty HFE; $\mathbf{1}$: the full HFE.
(3) E_m: the empty HFS; F_u: the full HFS.
(4) $\boldsymbol{H}$: the set of all HFEs; $\mathscr{H}$: the set of all HFSs on X.
(5) The decision maker's preference is risk-neutral, i.e., $\eta = \frac{1}{2}$.
(6) If $h_1 = \{\gamma_1^1, \gamma_2^1, \ldots, \gamma_l^1\}$ and $h_2 = \{\gamma_1^2, \gamma_2^2, \ldots, \gamma_l^2\}$ are two HFEs, then $h_1 \equiv h_2$ if and only if $\gamma_k^1 = \gamma_k^2$ $(k = 1, 2, \ldots, l)$.

□

2.2 Score Functions

Score function is usually recognized as a ranking tool for HFEs and HFSs by calculating the corresponding score values of them. Several HFE and HFS score functions with various forms have been introduced in the literature to compare HFEs (HFSs) and rank them [23, 28, 29, 31].

Farhadinia put forward the axiomatic definitions of HFE score function and HFS score function [31].

Definition 3 [31] Let $h_1 = \{\gamma_1^1, \gamma_2^1, \ldots, \gamma_l^1\}$ and $h_2 = \{\gamma_1^2, \gamma_2^2, \ldots, \gamma_l^2\}$ be two HFEs in $\boldsymbol{H}$. An HFE score function $S(\cdot): \boldsymbol{H} \to [0, 1]$ should satisfy the following two properties:

(1) *(monotone non-decreasing property)* if $\gamma_k^1 \le \gamma_k^2$ $(k = 1, 2, \ldots, l)$, then $S(h_1) \le S(h_2)$;
(2) *(boundary conditions property)* $S(\mathbf{0}) = 0$ and $S(\mathbf{1}) = 1$.

Definition 4 [31] Suppose $X = \{x_1, x_2, \ldots, x_n\}$ is the universe of discourse and $S(\cdot)$ an HFE score function. Let $H_1 = \{ <x_1, h_1(x_1)>, <x_2, h_1(x_2)>, \ldots, <x_n, h_1(x_n)> \}$ and $H_2 = \{ <x_1, h_2(x_1)>, <x_2, h_2(x_2)>, \ldots, <x_n, h_2(x_n)> \}$ be two HFSs in $\mathscr{H}$. An HFS score function $\mathbb{S}(\cdot): \mathscr{H} \to [0, 1]$ should satisfy the following two properties:

(1) *(monotone non-decreasing property)* if $S(h_1(x_i)) \le S(h_2(x_i))$ $(i = 1, 2, \ldots, n)$, then $\mathbb{S}(H_1) \le \mathbb{S}(H_2)$;
(2) *(boundary conditions property)* $\mathbb{S}(E_m) = 0$ and $\mathbb{S}(F_u) = 1$.

Some representative formulas of HFE and HFS score functions are displayed for the later analysis. Let $h = \{\gamma_1, \gamma_2, \ldots, \gamma_l\}$ be a hesitant fuzzy element.

Xia and Xu [23] defined an HFE score function as

$$S_X(h) = \frac{1}{l}\sum_{i=1}^{l}\gamma_i. \tag{2}$$

Farhadinia [29] presented an improved HFE score function as

$$S_F(h) = \frac{\sum_{i=1}^{l}\beta(i)\gamma_i}{\sum_{j=1}^{l}\beta(j)}, \tag{3}$$

where $\beta(i)$ $(i = 1, 2, \ldots, l)$ is a positive-valued monotonically decreasing sequence of index i. Later, Farhadinia [31] proposed a series of HFE score functions as follows:

The smallest score function

$$S_{\nabla}(h) = \begin{cases} 1, & \text{if } h = \mathbf{1}, \\ 0, & \text{otherwise}. \end{cases} \quad (4)$$

The greatest score function

$$S_{\Delta}(h) = \begin{cases} 0, & \text{if } h = \mathbf{0}, \\ 1, & \text{otherwise}. \end{cases} \quad (5)$$

The geometric-mean score function

$$S_{GM}(h) = \left(\prod_{k=1}^{l} \gamma_k \right)^{\frac{1}{l}}. \quad (6)$$

The product score function

$$S_P(h) = \prod_{k=1}^{l} \gamma_k. \quad (7)$$

The minimum score function and the maximum score function

$$S_{\min}(h) = \min\{\gamma_1, \gamma_2, \ldots, \gamma_l\} = \gamma_l, \quad (8a)$$

$$S_{\max}(h) = \max\{\gamma_1, \gamma_2, \ldots, \gamma_l\} = \gamma_1. \quad (8b)$$

The k-order statistic score function

$$S_{\sigma(k)}(h) = \gamma^{\sigma(k)}, \quad (9)$$

where $\gamma^{\sigma(k)}$ is the kth largest value in h. Actually, $S_{\sigma(1)} = S_{\max}(h)$ and $S_{\sigma(l)} = S_{\min}(h)$.

The fractional score function

$$S_{FF}(h) = \frac{\prod_{k=1}^{l} \gamma_k}{\prod_{k=1}^{l} \gamma_k + \prod_{k=1}^{l} (1 - \gamma_k)}, \quad (10)$$

with the convention $\frac{0}{0} = 0$.

Zhang and Xu [28] proposed a kind of HFE score function as

$$S_Z(h) = \left(\frac{\gamma_1^{\delta} + \gamma_2^{\delta} + \cdots + \gamma_l^{\delta}}{l} \right)^{\frac{1}{\delta}}, \quad (11)$$

where δ $(0 < \delta \leq 1)$ is a parameter to be tuned by decision makers.

The score value-based ranking rules are concluded as $h_1 \succ_{S(\cdot)} h_2$ if and only if $S_{(\cdot)}(h_1) > S_{(\cdot)}(h_2)$; $h_1 \sim_{S_{(\cdot)}} h_2$ if and only if $S_{(\cdot)}(h_1) = S_{(\cdot)}(h_2)$. The notation $h_1 \succ_{S(\cdot)} h_2$ means that h_1 is preferred to h_2 under the score function $S_{(\cdot)}$; $h_1 \sim_{S(\cdot)} h_2$ means that h_1 is identical to h_2 under $S_{(\cdot)}$; $h_1 \succsim h_2$ means that h_1 is not inferior to h_2 under $S_{(\cdot)}$.

There are relatively more score functions for HFEs than those for HFSs. The followings are four score functions for HFSs proposed by Farhadinia [31]. Let $H = \{ <x_1, h(x_1)>, <x_2, h(x_2)>, \ldots, <x_n, h(x_n)> \}$ be a hesitant fuzzy set in $\mathscr{H}$ and $S(\cdot)$ be an HFE score function.

The arithmetic-mean score function of H is defined as

$$\mathbb{S}_{AM}(H) = \frac{1}{n} \sum_{i=1}^{n} S(h(x_i)). \quad (12)$$

The geometric-mean score function of H is defined as

$$\mathbb{S}_{GM}(H) = \left(\prod_{i=1}^{n} S(h(x_i)) \right)^{\frac{1}{n}}. \quad (13)$$

The minimum score function and the maximum score function of H are defined as

$$\mathbb{S}_{\min}(H) = \min\{S(h(x_1)), S(h(x_2)), \ldots, S(h(x_n))\}, \quad (14a)$$

$$\mathbb{S}_{\max}(H) = \max\{S(h(x_1)), S(h(x_2)), \ldots, S(h(x_n))\}. \quad (14b)$$

In Eqs. (12)–(14), the HFE score function $S(\cdot)$ can be chosen from the set of HFE score functions.

3 Deviation Degrees and Deviation Score Functions for Hesitant Fuzzy Elements

An HFE is a basic unit of an HFS. Score functions for HFEs play a fundamental role in ranking alternatives described by hesitant fuzzy information. Those proposed HFE score functions given in Subsect. 2.2 have been applied to the real-life scenario. In the following, we are to explore the nature of the different score functions. First, we define an axiomatic definition of HFE deviation degree. Based on which, we construct a pair of HFE deviation score functions by calculating the deviation degrees between an HFE and the two extreme HFEs. Further, it is proved that deviation score functions can degenerate into all the existing score functions, which illustrates that a score function is indeed a special deviation degree. Finally, a new ranking method is developed to rank HFEs much finer than the current ones.

3.1 Deviation Degrees for HFEs

In this subsection, an axiomatic definition of HFE deviation degree is proposed to enrich the study of the measurements for depicting the differences among HFEs with individual preferences.

Definition 5 Let $h_1 = \{\gamma_1^1, \gamma_2^1, \ldots, \gamma_l^1\}$, $h_2 = \{\gamma_1^2, \gamma_2^2, \ldots, \gamma_l^2\}$ and $h_3 = \{\gamma_1^3, \gamma_2^3, \ldots, \gamma_l^3\}$ be three hesitant fuzzy elements in $\boldsymbol{H}$. An HFE deviation degree is a real-valued function $D : \boldsymbol{H} \times \boldsymbol{H} \rightarrow [0, 1]$ satisfying the following axiomatic requirements:

(1) $0 \leq D(h_1, h_2) \leq 1$;
(2) $D(h_1, h_1) = 0$;
(3) $D(h_1, h_2) = D(h_2, h_1)$;

(4) $D(\mathbf{0}, \mathbf{1}) = 1$;

(5) if $\gamma_k^1 \leq \gamma_k^2 \leq \gamma_k^3$ or $\gamma_k^1 \geq \gamma_k^2 \geq \gamma_k^3 (k = 1, 2, \ldots, l)$, then $D(h_1, h_2) \leq D(h_1, h_3)$ and $D(h_2, h_3) \leq D(h_1, h_3)$.

Remark 2 **An HFE deviation degree** is different from **an HFE distance.** In the axiomatical definition of distance measure d [5–7], it is demanded that $d(h_1, h_2) = 0 \Leftrightarrow h_1 \equiv h_2$. However, the condition $D(h_1, h_1) = 0$, i.e., $h_1 \equiv h_2 \Rightarrow D(h_1, h_2) = 0$, is required in the axiomatic definition of deviation degree. It is easy to see that $D(h_1, h_1) = 0$ is weaker than $d(h_1, h_2) \Leftrightarrow h_1 \equiv h_2$. So **an HFE distance must be an HFE deviation degree**, while **an HFE deviation degree might not be an HFE distance**. □

After the axiomatic definition of HFE deviation degree, we can produce a kind of HFE deviation degree with a special function and an aggregating operator.

Proposition 1 *Suppose* $h_1 = \{\gamma_1^1, \gamma_2^1, \ldots, \gamma_l^1\}$ *and* $h_2 = \{\gamma_1^2, \gamma_2^2, \ldots, \gamma_l^2\}$ *are two HFEs in* $\boldsymbol{H}$. *Let* $f : [0, 1] \to [0, 1]$ *be a monotonically increasing function with the properties that* $f(0) = 0$ *and* $f(1) = 1$. *Let* $\varDelta : [0, 1]^l \to [0, 1]$ *be an aggregating operator. Then,*

$$D^{(f,\varDelta)}(h_1,h_2) = \varDelta\left(f(|\gamma_1^1 - \gamma_1^2|), f(|\gamma_2^1 - \gamma_2^2|), \ldots, f(|\gamma_l^1 - \gamma_l^2|)\right) \quad (h_1, h_2 \in \boldsymbol{H}) \tag{15}$$

is an HFE deviation degree.

Proof

(1) Since $|\gamma_k^1 - \gamma_k^2| \in [0, 1]$ and $f : [0, 1] \to [0, 1]$, we have $f(|\gamma_k^1 - \gamma_k^2|) \in [0, 1]$. Hence, $D^{(f,\varDelta)}(h_1, h_2) = \varDelta(f(|\gamma_1^1 - \gamma_1^2|), f(|\gamma_2^1 - \gamma_2^2|), \ldots, f(|\gamma_l^1 - \gamma_l^2|)) \in [0, 1]$.

(2) It is easy to see $|\gamma_k^1 - \gamma_k^1| = 0$ $(k = 1, 2, \ldots, l)$, $f(0) = 0$ and $\varDelta(0, 0, \ldots, 0) = 0$. Hence, $D^{(f,\varDelta)}(h_1, h_1) = \varDelta(f(|\gamma_1^1 - \gamma_1^1|), f(|\gamma_2^1 - \gamma_2^1|), \ldots, f(|\gamma_l^1 - \gamma_l^1|)) = \varDelta(0, 0, \ldots, 0) = 0$.

(3) Since $|\gamma_k^1 - \gamma_k^2| = |\gamma_k^2 - \gamma_k^1|$ $(k = 1, 2, \ldots, l)$, it follows that $D^{(f,\varDelta)}(h_1, h_2) = D^{(f,\varDelta)}(h_2, h_1)$.

(4) From Eq. (15), we have that $D^{(f,\varDelta)}(\mathbf{0}, \mathbf{1}) = \varDelta(f(|1 - 0|), f(|1 - 0|), \ldots, f(|1 - 0|)) = \varDelta\ (1, 1, \ldots, 1) = 1$.

(5) Let $h_3 = \{\gamma_1^3, \gamma_2^3, \ldots, \gamma_l^3\}$. If $\gamma_k^1 \leq \gamma_k^2 \leq \gamma_k^3$ or $\gamma_k^1 \geq \gamma_k^2 \geq \gamma_k^3$, then we obtain $f(|\gamma_k^1 - \gamma_k^2|) \leq f(|\gamma_k^1 - \gamma_k^3|)$ $(k = 1, 2, \ldots, l)$ from the monotone increasing property of f. Note that $\varDelta$ is monotonic, thus $\varDelta(f(|\gamma_1^1 - \gamma_1^2|), f(|\gamma_2^1 - \gamma_2^2|), \ldots, f(|\gamma_l^1 - \gamma_l^2|)) \leq \varDelta(f(|\gamma_1^1 - \gamma_1^3|), f(|\gamma_2^1 - \gamma_2^3|), \ldots, f(|\gamma_l^1 - \gamma_l^3|))$, i.e., $D^{(f,\varDelta)}(h_1, h_2) \leq D^{(f,\varDelta)}(h_1, h_3)$. The proof of $D^{(f,\varDelta)}(h_2, h_3) \leq D^{(f,\varDelta)}(h_1, h_3)$ follows in a similar manner. □

We call $D^{(f,\varDelta)}(h_1, h_2)$ an $(f, \varDelta)$-related deviation degree, of which f is as an effect function. The effect function f and the aggregating operator $\varDelta$ can be set flexibly according to the needs.

Remark 3 The role of the function f is to enhance and weaken the differences between the two membership values in the corresponding position. $\varDelta$ is an aggregating operator, which reflects the attitude to the interactivities among the differences of the membership values. $\varDelta$ could be any one of these operators: max operator, min operator, AM (arithmetic-mean operator), WAM (weighted arithmetic-mean operator), OWA (ordered weighted average operator), GM (geometric-mean operator), WGM (weighted geometric-mean operator), and PW (prioritized aggregating operators) et al. [32, 33]. □

Specializing the effect function f and the aggregating operator $\varDelta$, we can obtain some concrete deviation degree formulas for HFEs.

Let $f(x) = x^2$ and $\varDelta$ =AM. Then,

$$D^{(x^2,\mathrm{AM})}(h_1, h_2) = \frac{1}{l}\sum_{k=1}^{l}\left(\gamma_k^1 - \gamma_k\right)^2. \tag{16}$$

Let $f(x) = x^{\frac{1}{2}}$ and $\varDelta$ =WAM. Then,

$$D^{(x^{\frac{1}{2}},\mathrm{WAM})}(h_1, h_2) = \sum_{k=1}^{l}\omega_k\left|\gamma_k^1 - \gamma_k^2\right|^{\frac{1}{2}}, \tag{17}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_l)$ is a normalized weight vector.

Let $f(x) = x^\delta$ $(\delta \neq 0)$ and $\varDelta$ = GM. Then,

$$D^{(x^\delta,\mathrm{GM})}(h_1, h_2) = \left(\prod_{k=1}^{l}\left|\gamma_k^1 - \gamma_k^2\right|^\delta\right)^{\frac{1}{l}}. \tag{18}$$

Remark 4 It is easy to prove that Eq. (18) is a deviation degree but not a distance measure. $D^{(f,\mathrm{GM})}$ with the other effect functions is not a distance measure. □

Example 2 Let $h_1 = \{0.7, 0.6, 0.5\}$ and $h_2 = \{1.0, 0.4, 0.3\}$ be two HFEs. Let $f = x$ and $\varDelta$ = AM. We have $D^{(x,\mathrm{AM})}(h_1, h_2) = \frac{1}{3} \times (|0.5 - 0.3| + |0.6 - 0.4| + |0.7 - 1.0|) = 0.23$. Let $f = x^2$ and $\varDelta$ = GM. We have $D^{(x^2,\mathrm{GM})}(h_1, h_2) = (|0.5 - 0.3|^2 \cdot |0.6 - 0.4|^2 \cdot |0.7 - 1.0|^2)^{\frac{1}{3}} = 0.05$.

Example 2 shows that the concrete HFE deviation degrees can be obtained by setting different effect functions and different aggregating operators in Eq. 15.

3.2 Deviation Score Functions for HFEs

In this subsection, we analyze the essence of HFE score functions from the deviation degree perspective.

The empty HFE, $\mathbf{0}$, is a special element in $\mathbf{H}$. There is no doubt that $\mathbf{0}$ is the smallest element in $\mathbf{H}$, and it is natural and reasonable to construct a score function of h by computing the deviation degree between h and $\mathbf{0}$.

Proposition 2 *Let $h = \{\gamma_1, \gamma_2, \ldots, \gamma_l\}$ be a hesitant fuzzy element in $\boldsymbol{H}$ and D a deviation degree for HFEs. Then,*

$$DS(h) = D(h, \mathbf{0}) \ (h \in \boldsymbol{H}) \tag{19}$$

is an HFE score function.

Proof We know from Def. 3 that a deviation score function should satisfy the axiom conditions.

(1) Let two HFEs $h_1 = \{\gamma_1^1, \gamma_2^1, \ldots, \gamma_l^1\}$ and $h_2 = \{\gamma_1^2, \gamma_2^2, \ldots, \gamma_l^2\}$ satisfy $0 \le \gamma_k^1 \le \gamma_k^2$ $(k = 1, 2, \ldots, l)$. By the requirement (5) in Definition 5, we obtain $D(h_1, \mathbf{0}) \le D(h_2, \mathbf{0})$, i.e., $DS(h_1) \le DS(h_2)$. Therefore, $DS(h)$ satisfies the monotone non-decreasing property.

(2) By the requirement (2) in Definition 5, we have $D(\mathbf{0}, \mathbf{0}) = 0$, i.e., $DS(\mathbf{0}) = 0$; by the requirements (3) and (4) in Definition 5, we obtain $D(\mathbf{1}, \mathbf{0}) = D(\mathbf{0}, \mathbf{1}) = 1$, i.e., $DS(\mathbf{1}) = 1$. Thus, the boundary conditions are satisfied. □

It is no doubt that the full HFE $\mathbf{1}$ is the biggest element in $\mathbf{H}$. Therefore, it is also reasonable to construct an HFE score function by computing the complementary of the deviation degree between h and $\mathbf{1}$.

Proposition 3 *Let $h = \{\gamma_1, \gamma_2, \ldots, \gamma_l\}$ be a hesitant fuzzy element in $\boldsymbol{H}$ and D a deviation degree for HFEs. Then,*

$$\overline{DS}(h) = 1 - D(h, \mathbf{1}) \ (h \in \boldsymbol{H}) \tag{20}$$

is an HFE score function.

Proof (1) Let two HFEs $h_1 = \{\gamma_1^1, \gamma_2^1, \ldots, \gamma_l^1\}$ and $h_2 = \{\gamma_1^2, \gamma_2^2, \ldots, \gamma_l^2\}$ satisfy $\gamma_k^1 \le \gamma_k^2 \le 1$ $(k = 1, 2, \ldots, l)$. By the requirement (5) in Definition 5, we obtain $D(h_1, \mathbf{1}) \ge D(h_2, \mathbf{1})$, then $1 - D(h_1, \mathbf{1}) \le 1 - D(h_2, \mathbf{1})$, i.e., $\overline{DS}(h_1) \le \overline{DS}(h_2)$. Therefore, $\overline{DS}(h)$ satisfies the monotone non-decreasing property.

(2) By the requirement (4) in Definition 5, we have $D(\mathbf{0}, \mathbf{1}) = 1$, then $\overline{DS}(\mathbf{0}) = 1 - D(\mathbf{0}, \mathbf{1}) = 1 - 1 = 0$. By the requirement (2) in Definition 5, we obtain $D(\mathbf{1}, \mathbf{1}) = 0$. Hence, $\overline{DS}(\mathbf{1}) = 1 - D(\mathbf{1}, \mathbf{1}) = 1 - 0 = 1$. Thus, the boundary conditions are satisfied. □

Both $DS(h)$ and $\overline{DS}(h)$ are called HFE deviation score functions.

Let $h = \{\gamma_1, \gamma_2, \ldots, \gamma_l\}$ be a hesitant fuzzy element in $\boldsymbol{H}$ and $D^{(f,\Delta)}$ an (f, Δ)-related HFE deviation degree.

$$\begin{aligned} DS^{(f,\Delta)}(h) &= D^{(f,\Delta)}(h, \mathbf{0}) \\ &= \Delta(f(|\gamma_1 - 0|), f(|\gamma_2 - 0|), \ldots, f(|\gamma_l - 0|)) \\ &= \Delta(f(\gamma_1), f(\gamma_2), \ldots, f(\gamma_l)) \end{aligned}$$

and

$$\begin{aligned} \overline{DS}^{(f,\Delta)}(h) &= 1 - D^{(f,\Delta)}(h, \mathbf{1}) \\ &= 1 - \Delta(f(|\gamma_1 - 1|), f(|\gamma_2 - 1|), \ldots, f(|\gamma_l - 1|)) \\ &= 1 - \Delta(f(1 - \gamma_1), f(1 - \gamma_2), \ldots, f(1 - \gamma_l)). \end{aligned}$$

Then, we call ($DS^{(f,\Delta)}, \overline{DS}^{(f,\Delta)}$) a pair of dual HFE deviation score functions with respect to $D^{(f,\Delta)}$.

Given an HFE deviation degree D, the value of $DS(h)$ is just the deviation degree between h and the negative ideal element $\mathbf{0}$, and the value of $\overline{DS}(h)$ is the complementary of the deviation degree between h and the positive ideal element $\mathbf{1}$. Inspired by the TOPSIS method, we call $DS(h)$ and $\overline{DS}(h)$ the negative HFE deviation score function (NHFEDSF) and the positive HFE deviation score function (PHFEDSF) to D, respectively.

Definition 6 An HFE deviation score function DS is self-dual iff $DS(h) = \overline{DS}(h)$ for any $h \in \mathbf{H}$.

A self-dual HFE deviation score function can be obtained by setting $DS^{(f,\Delta)}$ with special parameters. The following proposition provides a way for constructing a self-dual deviation score function. The score function is indeed the generalized form of the score function proposed by Xia and Xu in [23].

Proposition 4 *If f is an identity function and Δ is a normalized weighted arithmetic-mean operator, then $DS^{(f,\Delta)}$ is self-dual.*

Proof Given a hesitant fuzzy element $h = \{\gamma_1, \gamma_2, \ldots, \gamma_l\}$. A normalized weighted arithmetic-mean operator is denoted as $\Delta = \{\lambda_1, \lambda_2, \ldots, \lambda_l\}(\sum_{i=1}^{l} \lambda_i = 1)$. We have

$$\begin{aligned} \overline{DS}^{(f,\Delta)}(h) &= 1 - \Delta(f(1 - \gamma_1), f(1 - \gamma_2), \ldots, f(1 - \gamma_l)) \\ &= 1 - \sum_{i=1}^{l} \lambda_i (1 - \gamma_i) \\ &= 1 - \sum_{i=1}^{l} \lambda_i + \sum_{i=1}^{l} \lambda_i \gamma_i \\ &= \sum_{i=1}^{l} \lambda_i \gamma_i \\ &= \Delta(f(\gamma_1), f(\gamma_2), \ldots, f(\gamma_l)) \\ &= DS^{(f,\Delta)}(h). \end{aligned}$$

□

Let $f = x$ and $\varDelta = \{\frac{1}{l}, \frac{1}{l}, \ldots, \frac{1}{l}\}$, we obtain a self-dual HFE deviation score function $DS^{(x,\mathrm{AM})}$ soon.

Different effect functions and different aggregating operators can construct different deviation degrees and then deviation score functions. In the following, we are to show that the HFE score functions in the existing literature are special cases of the HFE deviation score function by setting the different effect functions and aggregating operators.

Proposition 5 *Let $h = \{\gamma_1, \gamma_2, \ldots, \gamma_l\}$ be a hesitant fuzzy element. If*

$$f_1(x) = \begin{cases} 1, & \text{if } 0 < x \leq 1, \\ 0, & \text{if } x = 0, \end{cases}$$

then

$$S_{\triangle}(h) = DS^{(f_1,\max)}(h).$$

Proof

(1) If $h = \mathbf{0}$, then

$$\begin{aligned} DS^{(f_1,\max)}(h) &= \max\{f_1(0-0), f_1(0-0), \ldots, f_1(0-0)\} \\ &= \max\{f_1(0), f_1(0), \ldots, f_1(0)\} \\ &= 0 \\ &= S_{\triangle}(h). \end{aligned}$$

(2) If $h \neq \mathbf{0}$, then $\exists \gamma_k$ such that $0 < \gamma_k \leq 1$. Since $f_1(\gamma_k) = 1$, it follows $DS^{(f_1,\max)}(h) = \max\{f_1(\gamma_1), f_1(\gamma_2), \ldots, f_1(\gamma_l)\} = 1$. Hence, $DS^{(f_1,\max)}(h) = 1 = S_{\triangle}(h)$. □

Proposition 6 *Let $h = \{\gamma_1, \gamma_2, \ldots, \gamma_l\}$ be a hesitant fuzzy element. If*

$$f_2(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } x = 1, \end{cases}$$

then

$$S_{\nabla}(h) = DS^{(f_2,\min)}(h).$$

Proof

(1) If $h = \mathbf{1}$, then

$$\begin{aligned} DS^{(f_2,\min)}(h) &= \min\{f_2(1-0), f_2(1-0), \ldots, f_2(1-0)\} \\ &= \min\{f_2(1), f_2(1), \ldots, f_2(1)\} \\ &= 1 \\ &= S_{\nabla}(h). \end{aligned}$$

(2) If $h \neq \mathbf{1}$, then $\exists \gamma_k$ such that $0 \leq \gamma_k < 1$. Since $f_2(\gamma_k) = 0$, it follows $DS^{(f_2,\min)}(h) = \min\{f_2(\gamma_1), f_2(\gamma_2), \ldots, f_2(\gamma_l)\} = 0$. Thus, $DS^{(f_2,\min)}(h) = 0 = S_{\nabla}(h)$. □

From the above two propositions, it is not difficult to see that the HFE deviation score function can degenerate to the two extreme score functions $S_{\triangle}(h)$ and $S_{\nabla}(h)$ with the extreme effect functions and the extreme aggregating operators.

Proposition 7 *Let $h = \{\gamma_1, \gamma_2, \ldots, \gamma_l\}$ be a hesitant fuzzy element. Then,*

$$S_{\triangle}(h) = \overline{DS}^{(f_2,\min)}(h)$$

and

$$S_{\nabla}(h) = \overline{DS}^{(f_1,\max)}(h).$$

Proof According to the definition of deviation score function, we have

$$\begin{aligned} &\overline{DS}^{(f_2,\min)}(h) \\ &= 1 - D^{(f_2,\min)}(h, 1) \\ &= 1 - \min\{f_2(1-\gamma_1), f_2(1-\gamma_2), \ldots, f_2(1-\gamma_l),\} \\ &= max\{1 - f_2(1-\gamma_1), 1 - f_2(1-\gamma_2), \ldots, 1 - f_2(1-\gamma_l)\} \\ &= max\{f_1(\gamma_1), f_1(\gamma_2), \ldots, f_1(\gamma_l)\} \\ &= DS^{(f_1,\max)}(h) \\ &= S_{\triangle}(h). \end{aligned}$$

Similarly, we can conclude $S_{\nabla}(h) = \overline{DS}^{(f_1,\max)}(h)$. □

From the proof of Proposition 7, we can draw an interesting conclusion that $DS^{(f_2,\min)}(h)$ is the dual score function of $DS^{(f_1,\max)}(h)$, i.e., $S_{\nabla}(h)$ is the dual score function of $S_{\triangle}(h)$. It coincides that f_1 is the complementary of f_2 and max is the dual operator of min. The operator f_1 matches with the operator max, and the operator f_2 matches with min in the definition of $DS(\cdot)$. By contrast, f_1 matches with min and f_2 matches with max in the definition of $\overline{DS}(\cdot)$.

Proposition 8 *Let $h = \{\gamma_1, \gamma_2, \ldots, \gamma_l\}$ be a hesitant fuzzy element. If $f(x) = x$ and $\varDelta$ is an OWA operator with the position weight vector λ generated by the quantifier $Q(\frac{k}{l-1}, \frac{k}{l-1})$ [34], then*

$$S_{\sigma(k)}(h) = DS^{(x,OWA_{\lambda})}(h).$$

Proof According to the definition of the quantifier $Q(a, b)$ in Ref. [34],

$$Q\left(\frac{k}{l-1},\frac{k}{l-1}\right)=\begin{cases}0, & if\ 0\leq x<\frac{k}{l-1},\\ 1, & if\ \frac{k}{l-1}\leq x\leq 1.\end{cases}$$

The ordered weight vector is $\lambda=\{0,\ldots,0,1,0,\ldots,0\}$ such that the kth weight is 1 and the others are 0. Hence,

$$\begin{aligned}DS^{(x,\mathrm{OWA}_\lambda)}(h)=&\mathrm{OWA}_\lambda(\mathrm{f}(\gamma_1),\mathrm{f}(\gamma_2),\ldots,\mathrm{f}(\gamma_l))\\ =&\mathrm{OWA}_\lambda(\gamma_1,\gamma_2,\ldots,\gamma_l)\\ =&\gamma_k\\ =&S_{\sigma(k)}(h).\end{aligned}$$

□

Proposition 9 *Let $h=\{\gamma_1,\gamma_2,\ldots,\gamma_l\}$ be a hesitant fuzzy element. If $f(x)=x$, then*

(1) $S_X(h)=DS^{(x,\mathrm{AM})}(h)=\frac{1}{l}\sum_{i=1}^{l}\gamma_i$;

(2) $S_{\mathrm{GM}}(h)=(S_P(h))^{\frac{1}{l}}=DS^{(x,\mathrm{GM})}(h)=\left(\prod_{k=1}^{l}\gamma_k\right)^{\frac{1}{l}}$.

Proposition 10 *Let $h=\{\gamma_1,\gamma_2,\ldots,\gamma_l\}$ be a hesitant fuzzy element. If $f(x)=x$ and Δ is a WAM operator with the position weight $0<\beta(1)\leq\beta(2)\leq\cdots\leq\beta(l)$, then*

$$S_F(h)=DS^{(x,\mathrm{WAM}_\beta)}(h)=\frac{\sum_{i=1}^{l}\beta(i)\gamma_i}{\sum_{j=1}^{l}\beta(j)}.$$

Lemma 1 *If $0\leq a_1\leq b_1\leq 1$ and $0\leq a_2\leq b_2\leq 1$, then*

$$\frac{a_1a_2}{a_1a_2+(1-a_1)(1-a_2)}\leq\frac{b_1b_2}{b_1b_2+(1-b_1)(1-b_2)}. \tag{21}$$

Proof We proof the conclusion from the following aspects.

(1) $b_1b_2=0$. Let $b_1=0$. $0\leq a_1\leq b_1\leq 1$ follows $a_1=0$. Equation (21) holds. The same for $b_2=0$.

(2) $b_1b_2\neq 0$ and $a_1a_2=0$. It can be easily found that the left of Eq. (21) equals to 0 and the right does not. Equation (21) holds.

(3) $b_1b_2\neq 0$ and $a_1a_2\neq 0$. Because $a_1\leq b_1$ and $a_2\leq b_2$, both of the inequalities $a_1a_2\leq b_1b_2$ and $(1-b_1)(1-b_2)\leq(1-a_1)(1-a_2)$ are true. We have

$$\frac{(1-a_1)(1-a_2)}{a_1a_2}\geq\frac{(1-b_1)(1-b_2)}{b_1b_2}$$

and

$$1+\frac{(1-a_1)(1-a_2)}{a_1a_2}\geq 1+\frac{(1-b_1)(1-b_2)}{b_1b_2}.$$

Then,

$$\frac{1}{1+\frac{(1-a_1)(1-a_2)}{a_1a_2}}\leq\frac{1}{1+\frac{(1-b_1)(1-b_2)}{b_1b_2}}.$$

Hence,

$$\frac{a_1a_2}{a_1a_2+(1-a_1)(1-a_2)}\leq\frac{b_1b_2}{b_1b_2+(1-b_1)(1-b_2)}.$$

□

Proposition 11 *Let $h=\{\gamma_1,\gamma_2,\ldots,\gamma_l\}$ be a hesitant fuzzy element. If $f(x)=x$ and $\Delta=\frac{\prod_{i=1}^{l}\gamma_i}{\prod_{i=1}^{l}\gamma_i+\prod_{i=1}^{l}(1-\gamma_i}$) is an operator, then*

(1) $D^{(x,\Delta)}$ *is a deviation degree*;

(2) $S_{FF}(h)=D^{(x,\Delta)}(h,\mathbf{0})=DS^{(x,\Delta)}(h)=\frac{\prod_{k=1}^{l}\gamma_k}{\prod_{k=1}^{l}\gamma_k+\prod_{k=1}^{l}(1-\gamma_k)}$.

Proof (1) We first check whether the deviation degree $D^{(x,\Delta)}$ will satisfy the five axiom conditions given in Def. 5. The first four conditions are obvious. Here we prove the last one.

Let $h_1=\{\gamma_1^1,\gamma_2^1,\ldots,\gamma_l^1\}$, $h_2=\{\gamma_1^2,\gamma_2^2,\ldots,\gamma_l^2\}$ and $h_3=\{\gamma_1^3,\gamma_2^3,\ldots,\gamma_l^3\}$ be three hesitant fuzzy elements in **H**. Given $0\leq\gamma_k^1\leq\gamma_k^2\leq\gamma_k^3\leq 1(k=1,2,\ldots,l)$, we follow that $0\leq|\gamma_k^1-\gamma_k^2|\leq|\gamma_k^1-\gamma_k^3|\leq 1$. By Lemma 1, we have

$$\frac{\prod_{k=1}^{l}|\gamma_k^1-\gamma_k^2|}{\prod_{k=1}^{l}|\gamma_k^1-\gamma_k^2|+\prod_{k=1}^{l}(1-|\gamma_k^1-\gamma_k^2|)}\leq\frac{\prod_{k=1}^{l}|\gamma_k^1-\gamma_k^3|}{\prod_{k=1}^{l}|\gamma_k^1-\gamma_k^3|+\prod_{k=1}^{l}(1-|\gamma_k^1-\gamma_k^3|)},$$

so $D^{(x,\Delta)}(h_1,h_2)\leq D^{(x,\Delta)}(h_1,h_3)$. The proof of $1\geq\gamma_k^1\geq\gamma_k^2\geq\gamma_k^3\geq 0(k=1,2,\ldots,l)$ is similar; hence, it is omitted. Therefore, $D^{(x,\Delta)}$ is a deviation degree.

(2) Let $h=\{\gamma_1,\gamma_2,\ldots,\gamma_l\}$ be an HFE. It can be obtained by simply computation that

$$\begin{aligned}S_{FF}(h)&=\frac{\prod_{k=1}^{l}\gamma_k}{\prod_{k=1}^{l}\gamma_k+\prod_{k=1}^{l}(1-\gamma_k)}\\ &=\frac{\prod_{k=1}^{l}|\gamma_k-0|}{\prod_{k=1}^{l}|\gamma_k-0|+\prod_{k=1}^{l}(1-|\gamma_k-0|)}\\ &=D^{(x,\Delta)}(h,\mathbf{0})\\ &=DS^{(x,\Delta)}(h).\end{aligned}$$

□

Proposition 12 *Let $h=\{\gamma_1,\gamma_2,\ldots,\gamma_l\}$ be a hesitant fuzzy element. If $f(x)=x^\delta$ $(0<\delta\leq 1)$ and Δ is an AM operator, then*

(1) $D_1 = (D^{(x^\delta, AM)})^\delta$ *is an deviation degree;*

(2) $S_Z(h) = D_1(h, \mathbf{0}) = D_1 S(h) = \left(\frac{\gamma_1^\delta + \gamma_2^\delta + \cdots + \gamma_l^\delta}{l}\right)^{\frac{1}{\delta}}$.

Proof (1) Obviously, $D^{(x^\delta, \mathrm{AM})}$ is a deviation degree. If $0 < \delta \leq 1$, then $(D^{(x^\delta, \mathrm{AM})})^\delta$ is a power function of $D^{(x^\delta, \mathrm{AM})}$ with the monotonic increasing property. It can be proved to satisfy the five conditions in Def. 5. Therefore, $D_1 = (D^{(x^\delta, \mathrm{AM})})^\delta$ is an deviation degree.

(2) By the simply computation, we have that

$$\begin{aligned} S_Z(h) &= \left(\frac{\gamma_1^\delta + \gamma_2^\delta + \cdots + \gamma_l^\delta}{l}\right)^{\frac{1}{\delta}} \\ &= \left(\frac{|\gamma_1 - 0)^\delta + |\gamma_2 - 0|^\delta + \cdots + |\gamma_l - 0|^\delta}{l}\right)^{\frac{1}{\delta}} \\ &= D_1(h, \mathbf{0}) \\ &= D_1 S(h). \end{aligned}$$

□

Proposition 12 shows that some of the existing deviation degrees are the functions of the deviation degrees. Therefore, we can obtain much more deviation score functions from deviation degrees.

Table 1 lists the propositions given above to demonstrate the nature of the existing HFE score functions by the proposed deviation degrees and deviation score functions. It can be concluded that an HFE score function can be seen as a special HFE deviation degree between an HFE and **0**/**1**. Besides, an HFE deviation degree can induce two HFE deviation score functions. There exist infinite HFE deviation degrees, so innumerable HFE deviation score functions can be constructed for the application in practice.

3.3 A New Ranking Method Based on a Pair of Dual HFE Deviation Score Functions

The single score value-based ranking method is quite simple and convenient; however, as analyzed in Ref. [28], there may exist too many indiscernible elements with the same score values, which means the corresponding ranking method is rough to some extent. Take $S_\triangle$ for example, if $h_1 \neq 0$ and $h_2 \neq 0$, then $S_\triangle(h_1) = S_\triangle(h_2)$, i.e., $h_1 \sim_{S_\triangle} h_2$, which illustrates that the $S_\triangle$-based ranking method is too rough to rank the HFEs. So does S_∇. In the following, we propose a new ranking method based on a pair of dual HFE deviation score functions.

Method I: *A ranking method based on a pair of dual HFE deviation score functions.*

Let h_1 and h_2 be two HFEs in $\mathbf{H}$ and D be an HFE deviation degree. $(DS, \overline{DS})$ is a pair of dual HFE deviation score functions constructed from the deviation degree D. Then, we set a new ranking rule based on the corresponding dual score functions.

(1) If $DS(h_1) > DS(h_2)$, then $h_1 \succ_{DS} h_2$.
(2) If $DS(h_1) = DS(h_2)$, then,

  (a) if $\overline{DS}(h_1) > \overline{DS}(h_2)$, then $h_1 \succ_{DS} h_2$;
  (b) if $\overline{DS}(h_1) = \overline{DS}(h_2)$, then $h_1 \sim_{DS} h_2$;
  (c) if $\overline{DS}(h_1) < \overline{DS}(h_2)$, then $h_2 \succ_{DS} h_1$.

Given an HFE deviation degree D, we can obtain two HFE deviation score functions DS and $\overline{DS}$. If two HFEs are with the same score value under DS, then we continue to calculate the score values under $\overline{DS}$ to compare their magnitude. If the two values calculated from $\overline{DS}$ are also the same, there are no differences between the corresponding HFEs; Otherwise, they are comparable. Since a

Table 1 Constructing the current HFE score functions in the literature

Effect function	Aggregating operator	Deviation degree	Deviation score function	Concrete score function
$f_1(x) = \begin{cases} 1, & \text{if } 0 < x \leq 1, \\ 0, & \text{if } x = 0. \end{cases}$	Max	$D^{(f_1, \max)}$	$DS^{(f_1, \max)}$	$S_\triangle$ (See [31])
$f_2(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } x = 1. \end{cases}$	Min	$D^{(f_2, \min)}$	$DS^{(f_2, \min)}$	S_∇ (See [31])
$f(x) = x$	OWA_λ with $Q(\frac{k}{l-1}, \frac{k}{l-1})$	$D^{(x, \mathrm{OWA}_\lambda)}$	$DS^{(x, \mathrm{OWA}_\lambda)}$	$S_{\sigma(k)}$ (See [31])
$f(x) = x$	AM	$D^{(x, \mathrm{AM})}$	$DS^{(x, \mathrm{AM})}$	S_X (See [23])
$f(x) = x$	GM	$D^{(x, \mathrm{GM})}$	$DS^{(x, \mathrm{GM})}$	S_{GM} (See [31])
$f(x) = x$	WAM	$D^{(x, \mathrm{WAM})}$	$DS^{(x, \mathrm{WAM})}$	S_F (See [29])
$f(x) = x$	$\varDelta(\gamma_1, \ldots, \gamma_l) = \frac{\prod_{i=1}^l \gamma_i}{\prod_{i=1}^l \gamma_i + \prod_{i=1}^l (1-\gamma_i)}$	$D^{(x, \varDelta)}$	$DS^{(x, \varDelta)}$	S_{FF} (See [31])
$f(x) = x^\delta$	AM	$D_1 = (D^{(x^\delta, AM)})^{\frac{1}{\delta}}$ $(\delta > 0)$	$D_1 S$	S_Z (See [28])

pair of dual HFE score functions are considered in the proposed method, the ranking results are usually much finer than those obtained by the single score value, which is true except for the self-dual deviation score functions. The following example is given to demonstrate the superiority of the proposed ranking method.

Example 3 Suppose $h_1 = \{0.7, 0.6, 0.5\}$, $h_2 = \{1.0, 0.3, 0.2\}$, $h_3 = \{0.8, 0.6, 0.4\}$, $h_4 = \{0.90, 0.5, 0.3\}$, and $h_5 = \{0.64, 0.6, 0.5\}$ are five HFEs in **H**. Apply Method I with the deviation degree $DS^{(x,\mathrm{GM})}$ to rank the five HFEs.

It is obvious that $DS^{(x,\mathrm{GM})} = S_{\mathrm{GM}}$. By using S_{GM} only, we obtain that $S_{\mathrm{GM}}(h_1) = 0.60$, $S_{\mathrm{GM}}(h_2) = 0.40$, $S_{\mathrm{GM}}(h_3) = 0.58$, $S_{\mathrm{GM}}(h_4) = 0.51$, and $S_{\mathrm{GM}}(h_5) = 0.58$. Then, we get the ranking of the five HFEs as

$$h_1 \succ_{S_{\mathrm{GM}}} \begin{pmatrix} h_3 \\ h_5 \end{pmatrix} \succ_{S_{\mathrm{GM}}} h_4 \succ_{S_{\mathrm{GM}}} h_2.$$

Since h_3 and h_5 are tied under S_{GM}, we continue to compute $\overline{DS}^{(x,\mathrm{GM})}(h_3) = 0.64$ and $\overline{DS}^{(x,\mathrm{GM})}(h_5) = 0.59$. Since $\overline{DS}^{(x,\mathrm{GM})}(h_3) > \overline{DS}^{(x,\mathrm{GM}}(h_5)$, it follows $h_3 \succ_{D^{(x,\mathrm{GM})}} h_5$. Thus, we have

$$h_1 \succ_{DS^{(x,\mathrm{GM})}} h_3 \succ_{DS^{(x,\mathrm{GM})}} h_5 \succ_{DS^{(x,\mathrm{GM})}} h_4 \succ_{DS^{(x,\mathrm{GM})}} h_2.$$

In the calculation processes above, we find that h_3 is with the same score value to h_5 by the score function S_{GM}, while h_3 is greater than h_5 by the dual deviation score functions. Most of HFE score functions are not self-dual, so the ranking results obtained by the proposed method are often superior to the ones by the single HFE score function-based approaches.

4 Deviation Degrees and Deviation Score Functions for Hesitant Fuzzy Sets

In this section, we are to introduce the concepts of deviation degree and deviation score function for HFSs as analogous to those for HFEs, and then develop a ranking method based on the dual HFS deviation score functions for ranking HFSs more efficiently.

4.1 Deviation Degrees for HFSs

In this subsection, we give the axiomatic definition of HFS deviation degree.

Definition 7 Suppose $X = \{x_1, x_2, \ldots, x_n\}$ is the universe of discourse. Let $H_1 = \{<x_1, h_1(x_1)>, <x_2, h_1(x_2)>, \ldots, <x_n, h_1(x_n)>\}$, $H_2 = \{<x_1, h_2(x_1)>, <x_2, h_2(x_2)>, \ldots, <x_n, h_2(x_n)>\}$ and $H_3 = \{<x_1, h_3(x_1)>, <x_2, h_3(x_2)>, \ldots, <x_n, h_3(x_n)>\}$ be three HFSs in $\mathscr{H}$. Let $D(\cdot)$ be a deviation degree for HFEs. An HFS deviation degree related to $D(\cdot)$ is a real-valued function $\mathbb{D}: \mathscr{H} \times \mathscr{H} \to [0, 1]$ satisfying the following axiomatic requirements:

(1) $0 \le \mathbb{D}(H_1, H_2) \le 1$;
(2) $\mathbb{D}(H_1, H_1) = 0$;
(3) $\mathbb{D}(H_1, H_2) = \mathbb{D}(H_2, H_1)$;
(4) $\mathbb{D}(E_m, F_u) = 1$;
(5) if $h_1(x_i) \succsim_{DS(\cdot)} h_2(x_i) \succsim_{DS(\cdot)} h_3(x_i)$ or $h_3(x_i) \succsim_{DS(\cdot)} h_2(x_i) \succsim_{DS(\cdot)} h_1(x_i)$ $(i = 1, 2, \ldots, n)$, then $\mathbb{D}(H_1, H_2) \le \mathbb{D}(H_1, H_3)$ and $\mathbb{D}(H_2, H_3) \le \mathbb{D}(H_1, H_3)$.

Remark 5 Similar to the analysis of HFE deviation degree, the axiomatic definition of HFS deviation degree is different from that of Farhadinia's HFS distance [7]. The conditions in the former are relaxer than the latter. Therefore, **an HFS distance is an HFS deviation degree, while an HFS deviation degree might not be an HFS distance**. □

After the axiomatic definition of HFS deviation degree, we can produce a kind of HFS deviation degree as follows.

Proposition 13 *Let* $H_1 = \{<x_1, h_1(x_1)>, <x_2, h_1(x_2)>, \ldots, <x_n, h_1(x_n)>\}$ *and* $H_2 = \{<x_1, h_2(x_1)>, <x_2, h_2(x_2)>, \ldots, <x_n, h_2(x_n)>\}$ *be two HFSs on X. Let D be an HFE deviation degree and* Δ *an aggregating operator for real numbers, then*

$$\mathbb{D}^{(D,\Delta)}(H_1,H_2) = \Delta(D(h_1(x_1),h_2(x_1)),D(h_1(x_2),h_2(x_2)),\ldots,D(h_1(x_n),h_2(x_n))) \quad (H_1,H_2 \in \mathscr{H}) \tag{22}$$

is a (D, Δ)*-related HFS deviation degree.*

Proof It can be easily followed from the properties of the deviation degree of HFEs and the aggregation operators, so a detailed proof is omitted here. □

Let $D^{(f,\Delta_1)}$ be an (f, Δ_1)-related HFE deviation degree. We call $\mathbb{D}^{(D^{(f,\Delta_1)},\Delta)}$ a $(D^{(f,\Delta_1)}, \Delta)$-related HFS deviation degree.

4.2 Deviation Score Functions for HFSs

We can analogously construct the deviation score functions for HFSs by the HFS deviation degree $\mathbb{D}$. Obviously, E_m and F_u are the smallest and the biggest HFSs in $\mathscr{H}$. It is rational to define the HFS deviation score functions by computing the special deviation degrees between an HFS and E_m or F_u.

Proposition 14 *Let $H = \{<x_1, h(x_1)>, <x_2, h(x_2)>, \ldots, <x_n, h(x_n)>\}$ be an HFS on X, $D(\cdot)$ an HFE deviation degree and $\mathbb{D}$ an HFS deviation degree. Then,*

$$\mathbb{DS}(H) = \mathbb{D}(H, E_m) \ (H \in \mathscr{H}) \tag{23}$$

is an HFS score function.

Proof (1) Let $H_1 = \{<x_1, h_1(x_1)>, <x_2, h_1(x_2)>, \ldots, <x_n, h_1(x_n)>\}$ and $H_2 = \{<x_1, h_2(x_1)>, <x_2, h_2(x_2)>, \ldots, <x_n, h_2(x_n)>\}$ such that $h_1(x_i) \succsim_{D(\cdot)} h_2(x_i) \succsim_{D(\cdot)} \mathbf{0}$ $(i = 1, 2, \ldots, n)$. By the requirement (5) of Definition 7, we have $\mathbb{D}(H_1, E_m) \geq \mathbb{D}(H_2, E_m)$, i.e., $\mathbb{DS}(H_1) \geq \mathbb{DS}(H_2)$. Therefore, $\mathbb{DS}$ satisfies the monotone non-decreasing property.

(2) By the requirements (2) and (4), we obtain $\mathbb{DS}(E_m) = \mathbb{D}(E_m, E_m) = 0$ and $\mathbb{DS}(F_u) = \mathbb{D}(F_u, E_m) = 1$. Thus, $\mathbb{DS}$ satisfies the boundary conditions property. □

Proposition 15 *Let $H = \{<x_1, h(x_1)>, <x_2, h(x_2)>, \ldots, <x_n, h(x_n)>\}$ be an HFS on X. Let $D(\cdot)$ be a deviation degree of HFEs and $\mathbb{D}$ an HFS deviation degree. Then,*

$$\overline{\mathbb{DS}}(H) = 1 - \mathbb{D}(H, F_u) \ (H \in \mathscr{H}) \tag{24}$$

is an HFS score function.

Proof Similar to the proof of Proposition 14. □

Given an HFS $H = \{<x_1, h(x_1)>, <x_2, h(x_2)>, \ldots, <x_n, h(x_n)>\}$, let $D^{(f,\Delta_1)}$ be an HFE deviation degree. $\mathbb{DS}^{(D^{(f,\Delta_1)},\Delta)}$ is recognized as a $(D^{(f,\Delta_1)}, \Delta)$-related HFS deviation score function. By simply computation, we have

$$\mathbb{DS}^{(D^{(f,\Delta_1)},\Delta)}(H) = \Delta\Big(DS^{(f,\Delta_1)}(h(x_1)), \ldots, DS^{(f,\Delta_1)}(h(x_n))\Big) \tag{25}$$

and

$$\overline{\mathbb{DS}}^{(D^{(f,\Delta_1)},\Delta)}(H) = 1 - \Delta\Big(1 - \overline{DS}^{(f,\Delta_1)}(h(x_1)), \ldots, 1 - \overline{DS}^{(f,\Delta_1)}(h(x_n))\Big). \tag{26}$$

Then, we call $(\mathbb{DS}^{(D^{(f,\Delta_1)},\Delta)}, \overline{\mathbb{DS}}^{(D^{(f,\Delta_1)},\Delta)})$ a pair of dual HFS deviation score functions.

We know that each HFS deviation degree on $\mathscr{H}$ can generate two HFS score functions. The physical meaning of $\mathbb{DS}(H)$ is the deviation degree between H and E_m; $\overline{\mathbb{DS}}(H)$ is the complementary of the deviation degree between H and F_u. We also call $\mathbb{DS}(H)$ and $\overline{\mathbb{DS}}(H)$ the negative HFS deviation score function (NHFSDSF) and the positive HFS deviation score function (PHFSDSF) of H with respect to $\mathbb{D}$, respectively. There exist infinite HFS deviation degrees, so we can obtain countless HFS deviation score functions.

Definition 8 An HFS deviation score function $\mathbb{DS}$ is self-dual iff $\mathbb{DS}(h) = \overline{\mathbb{DS}}(h)$ for any $H \in \mathscr{H}$.

A self-dual HFS deviation score function can be obtained by setting $\mathbb{DS}^{(D^{(f,\Delta_1)}),\Delta)}$ with three special parameters f, Δ_1 and Δ. The following proposition provides a way to construct a self-dual HFS deviation score function.

Proposition 16 *Suppose $H = \{<x_1, h(x_1)>, <x_2, h(x_2)>, \ldots, <x_n, h(x_n)>\}$ with $h(x_j) = \{\gamma_1^j, \gamma_2^j, \ldots, \gamma_l^j\}$ is a hesitant fuzzy set*, and $\mathbb{DS}^{(D^{(f,\Delta_1)},\Delta)}$ *and $\overline{\mathbb{DS}}^{(D^{(f,\Delta_1)},\Delta)}$ are a pair of dual deviation score functions for HFSs. Let $f = x$, and let Δ_1 and Δ be two normalized weighted arithmetic-mean operators; then we have*

$$\mathbb{DS}^{(D^{(x,\Delta_1)},\Delta)}(H) = \overline{\mathbb{DS}}^{(D^{(x,\Delta_1)},\Delta)}(H).$$

Proof Suppose $\Delta_1 = \{\lambda_1, \lambda_2, \ldots, \lambda_l\}$ with $\sum_{i=1}^{l} \lambda_i = 1$ and $\Delta = \{\omega_1, \omega_2, \ldots, \omega_n\}$ with $\sum_{j=1}^{n} \omega_j = 1$. Then,

$$\begin{aligned}
&\overline{\mathbb{DS}}^{(D^{(x,\Delta_1)},\Delta)}(H)\\
&= 1 - \Delta\Big(1 - \overline{DS}^{D^{(x,\Delta_1)}}(h(x_1)), \ldots, 1 - \overline{DS}^{D^{(x,\Delta_1)}}(h(x_n))\Big)\\
&= 1 - \sum_{j=1}^{n}\Big(\omega_j\Big(1 - \overline{DS}^{D^{(x,\Delta_1)}}(h(x_j))\Big)\Big)\\
&= \sum_{j=1}^{n} \omega_j\Big(\overline{DS}^{D^{(x,\Delta_1)}}(h(x_j))\Big)\\
&= \sum_{j=1}^{n}\left(\omega_j\left(1 - \sum_{i=1}^{l} \lambda_i f(1 - \gamma_i^j)\right)\right)\\
&= \sum_{j=1}^{n} \omega_j\left(1 - \sum_{i=1}^{l} \lambda_i (1 - \gamma_i^j)\right)\\
&= \sum_{j=1}^{n} \omega_j \sum_{i=1}^{l} \lambda_i \gamma_i^j\\
&= \sum_{j=1}^{n} \omega_j DS^{(x,\Delta_1)}(h(x_j))\\
&= \mathbb{DS}^{(D^{(x,\Delta_1)},\Delta)}(H).
\end{aligned}$$

□

Given an HFS deviation score function $\mathbb{D}^{(D^{(f,\Delta_1)},\Delta)}$, specifying Δ in $\mathbb{D}^{(D^{(f,\Delta_1)},\Delta)}$ and staying HFE deviation degree $D^{(f,\Delta_1)}$ the same, we construct the following HFS score functions [7].

$$\mathbb{DS}^{(D^{(f,\Delta_1)},\mathrm{AM})}(H)=\frac{1}{n}\sum_{i=1}^{n}DS^{(f,\Delta_1)}(h(x_i)). \tag{27}$$

$$\mathbb{DS}^{(D^{(f,\Delta_1)},\mathrm{GM})}(H)=\left(\prod_{i=1}^{n}DS^{(f,\Delta_1)}(h(x_i))\right)^{\frac{1}{n}}. \tag{28}$$

$$\mathbb{DS}^{(D^{(f,\Delta_1)},\min)}(H)=\min\{DS^{(f,\Delta_1)}(h(x_1)),\ldots,DS^{(f,\Delta_1)}(h(x_n))\}, \tag{29a}$$

$$\mathbb{DS}^{(D^{(f,\Delta_1)},\max)}(H)=\max\{DS^{(f,\Delta_1)}(h(x_1)),\ldots,DS^{(f,\Delta_1)}(h(x_n))\}. \tag{29b}$$

The HFS deviation score functions listed above are corresponding to Eqs. (12)–(14) in 2.2.

4.3 A New Ranking Method Based on a Pair of Dual HFS Deviation Score Functions

HFS score functions provide decision makers with a non-aggregation way to solve HFMADM [31]. In analogy to the representation of the above HFS score functions in terms of HFE score functions, we propose a new method for ranking HFSs based on a pair of dual HFS deviation score functions.

Method II: *A ranking method based on a pair of dual HFS deviation score functions.*

Let H_1 and H_2 be two HFSs in $\mathscr{H}$ and $\mathbb{D}$ an HFS deviation degree. $(\mathbb{DS},\overline{\mathbb{DS}})$ is a pair of dual HFS deviation score functions.

(1) If $\mathbb{DS}(H_1)>\mathbb{DS}(H_2)$, then $H_1\succ_{\mathbb{DS}}H_2$.
(2) If $\mathbb{DS}(H_1)=\mathbb{DS}(H_2)$, then

 (a) if $\overline{\mathbb{DS}}(H_1)>\overline{\mathbb{DS}}(H_2)$, then $H_1\succ_{\mathbb{DS}}H_2$;
 (b) if $\overline{\mathbb{DS}}(H_1)=\overline{\mathbb{DS}}(H_2)$, then $H_1\sim_{\mathbb{DS}}H_2$;
 (c) if $\overline{\mathbb{DS}}(H_1)<\overline{\mathbb{DS}}(H_2)$, then $H_2\succ_{\mathbb{DS}}H_1$.

If the two HFSs are of the same value under $\mathbb{DS}$, then we continue to calculate the values under $\overline{\mathbb{DS}}$ for comparing their magnitude. If the values of the two HFSs are equal under both $\mathbb{DS}$ and $\overline{\mathbb{DS}}$, then there are no differences between the corresponding HFSs. Since the newly proposed method checks a pair of HFS deviation score values, the corresponding ranking results are always superior to those by way of the single score value-based approach in the literature. This statement is correct in the condition that the HFS deviation score functions are not self-dual. And it is worth noting that the self-dual deviation score function-based method is with the same ranking performance as the single score value-based one. The following example illustrates the superiority of the proposed ranking method above.

Example 4 Suppose $X=\{x_1,x_2\}$. $H_1=\{<x_1,\{0.8,0.6,0.4\}>,<x_2,\{0.3,0.2\}>\}$ and $H_2=\{<x_1,\{0.64,0.6,0.5\}>,<x_2,\{0.6,0.1\}>\}$ are two HFSs on X. Let $D^{(f,\mathrm{GM})}$ being an HFE deviation degree with $f=x$. Based on $D^{(x,\mathrm{GM})}$, we define an HFS deviation degree $\mathbb{D}^{(D^{(x,\mathrm{GM})},AM)}$ and an HFS deviation score function $\mathbb{DS}^{(D^{(x,\mathrm{GM})},\mathrm{AM})}$. Then, we have

$$\begin{aligned}&\mathbb{DS}^{(D^{(x,\mathrm{GM})},\mathrm{AM})}(H_1)\\&=\frac{1}{2}\times(DS^{(x,\mathrm{GM})}(\{0.8,0.6,0.4\})+DS^{(x,\mathrm{GM})}(\{0.3,0.2\}))\\&=\frac{1}{2}\times\left((0.8\times0.6\times0.4)^{\frac{1}{3}}+(0.3\times0.2)^{\frac{1}{2}}\right)\\&=0.41\end{aligned}$$

and

$$\begin{aligned}&\mathbb{DS}^{(D^{(x,\mathrm{GM})},\mathrm{AM})}(H_2)\\&=\frac{1}{2}\times(DS^{(x,\mathrm{GM})}(\{0.64,0.6,0.5\})+DS^{(x,\mathrm{GM})}(\{0.1,0.6\}))\\&=\frac{1}{2}\times\left((0.64\times0.6\times0.5)^{\frac{1}{3}}+(0.1\times0.6)^{\frac{1}{2}}\right)\\&=0.41.\end{aligned}$$

The two HFSs are tied under the score function $\mathbb{DS}^{(D^{(x,\mathrm{GM})},\mathrm{AM})}$. Now, we continue to compute

$$\begin{aligned}&\overline{\mathbb{DS}}^{(D^{(x,\mathrm{GM})},\mathrm{AM})}(H_1)\\&=1-\frac{1}{2}\times((1-\overline{DS}^{(x,\mathrm{GM})}(\{0.8,0.6,0.4\}))+\\&(1-\overline{DS}^{(x,\mathrm{GM})}(\{0.3,0.2\})))\\&=1-\frac{1}{2}\times\left((1-(0.2\times0.4\times0.6)^{\frac{1}{3}})+\left(1-(0.7\times0.8)^{\frac{1}{2}}\right)\right)\\&=1-0.444\\&=0.56\end{aligned}$$

and

$$\begin{aligned}&\overline{\mathbb{DS}}^{(D^{(x,GM)},AM)}(H_2)\\&=1-\frac{1}{2}\times((1-\overline{DS}^{(x,GM)}(\{0.64,0.6,0.5\}))+\\&(1-\overline{DS}^{(x,GM)}(\{0.1,0.6\})))\\&=1-\frac{1}{2}\times\left((1-(0.36\times0.4\times0.5)^{\frac{1}{3}})+\left(1-(0.9\times0.4)^{\frac{1}{2}}\right)\right)\\&=1-0.492\\&=0.51.\end{aligned}$$

Therefore, we have $H_1\succ_{\mathbb{DS}^{(D^{(x,\mathrm{GA})},\mathrm{AM})}}H_2$.

From Example 4, the values of the two HFSs are identical to each other by the single score value-based method, and they are discernible by the proposed HFS ranking method. Since the latter considers more HFS score information, the obtained results are usually better than the ones by the former.

5 Discussions About Deviation Score Functions

The above two sections have proposed a series of deviation score functions for HFEs and HFSs, which reveal the nature of the HFE and HFS score functions. We may get confused about how to choose the most appropriate one from the plentiful deviation score functions. Therefore, it is necessary to provide a guide for selecting a proper deviation score function. In this section, we are to solve the problem by analyzing the mathematical and behavioral properties of HFE and HFS deviation score functions.

Definition 9 [31] A score function is said to be symmetric if the score value does not depend on the order of its arguments. That is,

(1) for an HFE $h=\{\gamma_1,\gamma_2,\ldots,\gamma_l\}$, if $h'=\{\widetilde{\gamma}_1,\widetilde{\gamma}_2,\ldots,\widetilde{\gamma}_l\}$ and $(\widetilde{\gamma}_1,\widetilde{\gamma}_2,\ldots,\widetilde{\gamma}_l)$ is a permutation of $(\gamma_1,\gamma_2,\ldots,\gamma_l)$, then $S(h)=S(h')$;

(2) for an HFS $H=\{<x_1,h(x_1)>,<x_2,h(x_2)>,\ldots,<x_n,h(x_n)>\}$, if $H'=\{<\widetilde{x}_1,h(\widetilde{x}_1)>,<\widetilde{x}_2,h(\widetilde{x}_2)>,\ldots,<\widetilde{x}_n,h(\widetilde{x}_n)>\}$ and $(<\widetilde{x}_1,h(\widetilde{x}_1)>,<\widetilde{x}_2,h(\widetilde{x}_2)>,\ldots,<\widetilde{x}_n,h(\widetilde{x}_n)>)$ is a permutation of $(<x_1,h(x_1)>,<x_2,h(x_2)>,\ldots,<x_n,h(x_n)>)$, then $\mathbb{S}(H)=\mathbb{S}(H')$.

Proposition 17 *Let* $h=\{\gamma_1,\gamma_2,\ldots,\gamma_l\}$ *and* $D^{(f,\Delta)}$ *be an HFE deviation degree. Suppose* $DS^{(f,\Delta)}(h)$ *and* $\overline{DS}^{(f,\Delta)}(h)$ *are two HFE score functions with respect to* $D^{(f,\Delta)}$. *If* Δ *is a symmetric operating operator, then* $DS^{(f,\Delta)}(h)$ *and* $\overline{DS}^{(f,\Delta)}(h)$ *are symmetric HFE score functions.*

Proof Since Δ is symmetric, it follows that $DS^{(f,\Delta)}(h)=\Delta(f(\gamma_1),f(\gamma_2),\ldots,f(\gamma_l))$ does not depend on the order of memberships in h. Therefore, $DS^{(f,\Delta)}$ is a symmetric HFE score function. The proof for $\overline{DS}^{(f,\Delta)}$ is omitted here. □

Proposition 18 *Let* $H=\{<x_1,h(x_1)>,<x_2,h(x_2)>,\ldots,<x_n,h(x_n)>\}$ *and* $\mathbb{D}^{(D,\Delta)}$ *be a* (D,Δ)*-related HFS deviation degree. Suppose* $\mathbb{DS}^{(D,\Delta)}(H)$ *and* $\overline{\mathbb{DS}}^{(D,\Delta)}(H)$ *are two HFS score functions with respect to* $\mathbb{D}^{(D,\Delta)}$. *If* Δ *is a symmetric operating operator, then* $\mathbb{DS}^{(D,\Delta)}(H)$ *and* $\overline{\mathbb{DS}}^{(D,\Delta)}(H)$ *are symmetric HFS score functions.*

Proof If Δ is symmetric, then $\mathbb{DS}^{(D,\Delta)}(H)=\Delta(DS(h(x_1)),DS(h(x_2)),\ldots,DS(h(x_n)))$ does not depend on the order of HFEs in H. Therefore, $\mathbb{DS}^{(D,\Delta)}(H)$ is a symmetric HFS score function. The proof for $\overline{\mathbb{DS}}^{(D,\Delta)}(H)$ is omitted. □

For either HFE deviation score functions or HFS deviation score functions, if Δ is one of the operators: max, min, OWA, AM, GM, then the corresponding deviation score functions are all symmetric.

Definition 10 [31] An HFE score function is said to be idempotency if it satisfies the condition $S(h)=\gamma$ for $h=\{\gamma,\gamma,\ldots,\gamma\}$. An HFS score function is referred to as idempotency if it satisfies the condition $\mathbb{S}(H)=S(h)$ for $H=\{<x_1,h>,<x_2,h>,\ldots,<x_n,h>\}$.

Proposition 19 *Let* $DS^{(f,\Delta)}$ *and* $\overline{DS}^{(f,\Delta)}$ *be two dual HFE score functions. If* $f(x)=x$ *and* $\Delta\in\{\min,\max,\mathrm{AM},\mathrm{OWA},\mathrm{WAM},\mathrm{GM}\}$, *then* $DS^{(x,\Delta)}$ *and* $\overline{DS}^{(x,\Delta)}$ *are idempotency.*

Proof It can easily follow from the idempotency property of Δ. □

Definition 11 [31] Let $h=\{\gamma_1,\gamma_2,\ldots,\gamma_l\}$. An HFE score function S is said to be

(1) conjunctive, if $S(h)\le\min\{\gamma_i\ |i=1,2,\ldots,n\}$;
(2) disjunctive, if $S(h)\ge\max\{\gamma_i\ |i=1,2,\ldots,n\}$;
(3) internal, if $\min\{\gamma_i\ |i=1,2,\ldots,n\}\le S(h)\le\max\{\gamma_i\ |i=1,2,\ldots,n\}$;
(4) mixed, if $S(h)$ is neither conjunctive nor disjunctive nor internal.

Proposition 20 *Let* $f(x)=x$. *Then the HFE deviation score functions* $DS^{(f,\min)}$, $\overline{DS}^{(x,\max)}$ *and* $(DS^{(x,\mathrm{GM})})^l$ *are conjunctive;* $DS^{(x,\max)}$ *and* $\overline{DS}^{(x,\min)}$ *are disjunctive; and* $DS^{(x,\Delta)}$ *is internal.*

Proof For the boundary property of the aggregating operator, the above conclusions can be easily proved. □

Definition 12 Let $H=\{<x_1,h(x_1)>,<x_2,h(x_2)>,\ldots,<x_n,h(x_n)>\}$ and S be an HFE score function. An HFS score function $\mathbb{S}$ is said to be

(1) conjunctive, if $\mathbb{S}(H)\le\min\{S(h(x_i))|i=1,2,\ldots,n\}$;
(2) disjunctive, if $\mathbb{S}(H)\ge\max\{S(h(x_i))|i=1,2,\ldots,n\}$;
(3) internal, if $\min\{S(h(x_i))|i=1,2,\ldots,n\}\le\mathbb{S}(H)\le\max\{S(h(x_i))|i=1,2,\ldots,n\}$;
(4) mixed, if $\mathbb{S}(H)$ is neither conjunctive nor disjunctive nor internal.

Proposition 21 *Let* $H=\{<x_1,h(x_1)>,<x_2,h(x_2)>,\ldots,<x_n,h(x_n)>\}$ *and* $\mathbb{D}^{(D,\Delta)}$ *be a* (D,Δ)*-related HFS deviation degree.* $\mathbb{DS}^{(D,\min)}$, $\overline{\mathbb{DS}}^{(D,\max)}$ *and*

$(\mathbb{DS}^{(D,GM)})^n$ are conjunctive; $\mathbb{DS}^{(D,\max)}$ and $\overline{\mathbb{DS}}^{(D,\min)}$ are disjunctive; $\mathbb{DS}^{(D,\Delta)}$ is internal.

Proof The conclusions can be easily followed by the boundary property of the aggregating operator. It is omitted. □

Farhadinia [31] has discussed the rules for selecting HFE scores. In the following, we introduce the rules for choosing **HFS** deviation score functions.

(1) If the HFEs in an HFS are connected by a logical "and" relation, then the chosen HFS score function should be conjunctive, which means that the HFS score value will be high if and only if the score value of each HFE is high. The HFS deviation score function with the "min" operator is a typical conjunctive HFS score function.

(2) If the HFEs in an HFS are connected by a logical "or" relation, then the selected HFS score function should be disjunctive, which means that the HFS score value will be low if and only if the score value of each HFE is low. The HFS deviation score function with the "max" operator is a typical disjunctive HFS score function.

(3) If there exist compensative relations between HFEs, the chosen HFS score function should be internal, i.e., an HFE with a smaller score value can be compensated by an HFE with a higher score value. The HFS deviation score function with the "AM" and "WAM" operators are the typical internal HFS score functions.

(4) Mixed HFS score values sometimes are extended beyond the minimum and maximum score values of the HFEs in an HFS.

(5) It is worthy to mention that weighted HFS score functions are those structured by extension of the equal-weighted one. Weighted score functions can be used in most of the multi-attribute decision-making problems in which the attributes are addictive and independent. In the following section, this kind of HFS deviation score function will be used to solve a practical HFMADM problem.

The section has analyzed the mathematical and behavioral properties of HFE and HFS deviation score functions, and further provided suggestions for selecting appropriate HFS deviation score function from a variety of HFS deviation score functions.

6 The Application of HFS Deviation Score Functions in Multi-attribute Decision Making

Multi-attribute decision making addresses the problem of making an optimal choice with the highest degree of satisfaction from a set of alternatives in terms of several attributes. Due to the inherent vagueness of human preferences, the attributes involved in decision-making problems are often evaluated not by real numbers, but by fuzzy information. Especially in the face of the decision scenario such as earthquake early warning, transportation networks, online matching markets, and so on, the involved experts need to make real-time decisions in a dynamic environment, which is faster, more accurate, and more stable than conventional decisions [35]. In these real-time decision-making problems, experts have no time to carefully consider and evaluate each alternative under each attribute and reach an agreement through discussion. The classical fuzzy set with a unique membership for an object does not work quite well in this case; however, the hesitant fuzzy set using a finite set of values to describe membership for an object can model them better. HFEs and HFSs play a significant role in the flexible expression of expert evaluations, and the corresponding score functions have the characteristics of fast computability and high discernibility, which can work better than the conventional aggregation technologies under the real-time dynamic circumstance. This section starts to present the formal description of a multi-attribute decision-making problem in a hesitant fuzzy setting. To solve the multi-attribute decision-making problem through a way of non-aggregation, we propose an HFS deviation score value-based approach to rank the alternatives effectively. Finally, a case study concerning human resource management is employed to illustrate the advantages of the proposed method.

6.1 Description of HFMADM

Suppose there exist n alternatives which make up the alternative set $X = \{x_1, x_2, \ldots, x_n\}$. Each alternative is evaluated on an attribute set with m attributes as $A = \{a_1, a_2, \ldots, a_m\}$. We also suppose the attributes in A are independent, and then the weight vector of attributes are expressed as $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^{\mathrm{T}}$ such that $\omega_j \in [0, 1]$ and $\sum_{j=1}^{m} \omega_j = 1$. Suppose a group of decision makers anonymously provide all the possible degrees that the alternative x_i satisfies the attribute a_j, which can be denoted by a hesitant fuzzy element h_{ij}. The evaluations of the alternative x_i under all the attributes construct a hesitant fuzzy set $H_i = \{ < a_j, h_{ij} | \, a_j \in A > \} \; (i = 1, 2, \ldots, n)$.

The task of HFMADM considered in this study is to choose the best alternative(s) from the alternative set X.

6.2 An HFS Deviation Score Function-Based Method for Solving HFMADM Problem

We give a method based on HFS deviation score values for solving multi-attribute decision making with hesitant fuzzy information.

Method III: *An HFS deviation score function-based method for solving HFMADM problem.*

Decision makers provide the possible evaluations about the alternative x_i under the attribute a_j, which are denoted by h_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$.

Step 1. Identify the HFSs $H_i = \{<a_1, h_{i1}>, <a_1, h_{i2}>, \ldots, <a_i, h_{im}>\}$ corresponding to alternatives x_i $(i = 1, 2, \ldots, n)$.
Step 2. Select an HFS deviation degree $\mathbb{D}$, and simultaneously determine the NHFSDSF $\mathbb{DS}$ and PHFSDSF $\overline{\mathbb{DS}}$.
Step 3. Calculate $\mathbb{DS}(H_i)$, and $\overline{\mathbb{DS}}(H_i)$ $(i = 1, 2, \ldots, n)$ if needed.
Step 4. Rank the HFSs H_i with the score values $\mathbb{DS}(H_i)$ and $\overline{\mathbb{DS}}(H_i)$ by Method II.
Step 5. Select the best alternative(s).

Determining a concrete deviation score function $\mathbb{DS}^{D^{(f,\Delta_1)},\Delta)}$ is a critical step of Method III. In solving a practical HFMADM problem, Δ_1 lies on the attitude to the experts' evaluation values, and Δ on the relations among different attributes. Decision makers may specify the two operators in $\mathbb{DS}^{D^{(f,\Delta_1)},\Delta)}$ referring to the behavioral properties of HFE score functions [31] and those of HFS score functions proposed in Section 5. In many multi-attribute group decision-making problem, the operator AM is recommended for the equal importance of experts, and the operator WAM is advised for the compensative relations between attributes. In some other scenarios, decision makers might determine the HFS deviation score function with any other more suitable operators.

6.3 An Illustrative Example

Suppose that an HR manager of a software company desires to hire a system analysis engineer. After the initial professional experience screening, five alternatives x_1, x_2, x_3, x_4 , and x_5 have been retained for further evaluation. A committee of three decision makers has been formed to interview them and to select the most appropriate one. Three criteria, including emotional steadiness (a_1), personality (a_2), and oral communication skill (a_3), are considered, and the weight vector of attributes is $\omega = \{0.3, 0.4, 0.3\}$. For an alternative under an attribute, some decision makers do not provide their evaluation values for the avoidance of co-worker. Besides, a value repeated more times does not indicate that it has more importance than other values repeated fewer times [8]. Table 2 presents the hesitant fuzzy evaluation matrix.

Step 1. From Table 2, one can express the HFSs corresponding to x_i $(i = 1, 2, \ldots, 5)$ as follows:

Table 2 An evaluation system with hesitant fuzzy information

	a_1	a_2	a_3
x_1	{0.5, 0.4, 0.3}	{0.5, 0.4, 0.2}	{0.9, 0.6, 0.4}
x_2	{0.7, 0.3, 0.2}	{0.8, 0.6, 0.5}	{0.7, 0.4, 0.3}
x_3	{0.7, 0.6}	{0.9, 0.6}	{0.6, 0.4}
x_4	{0.6, 0.45}	{0.95, 0.5}	{0.7, 0.6}
x_5	{0.7, 0.3}	{0.8, 0.4}	{0.7, 0.5, 0.3}

$$H_1 = \{\{0.5, 0.4, 0.3\}, \{0.5, 0.4, 0.2\}, \{0.9, 0.6, 0.4\}\},$$
$$H_2 = \{\{0.7, 0.3, 0.2\}, \{0.8, 0.6, 0.5\}, \{0.7, 0.4, 0.3\}\},$$
$$H_3 = \{\{0.7, 0.6\}, \{0.9, 0.6\}, \{0.6, 0.4\}\},$$
$$H_4 = \{\{0.6, 0.45\}, \{0.95, 0.5\}, \{0.7, 0.6\}\},$$
$$H_5 = \{\{0.7, 0.3\}, \{0.8, 0.4\}, \{0.7, 0.5, 0.3\ gt\}.$$

Step 2. The decision maker chooses $\mathbb{DS}^{(D^{(x^2,\mathrm{AM})},\mathrm{WAM})}$ with WAM of (0.3, 0.4, 0.3). Then, the HFS deviation score function is given as

$$\mathbb{DS}^{(D^{(x^2,\mathrm{AM})},\mathrm{WAM})}(H_i) = 0.3 \times DS^{(x^2,\mathrm{AM})}(h_{i1}) + 0.4 \times DS^{(x^2,\mathrm{AM})}(h_{i2}) + 0.3 \times DS^{(x^2,\mathrm{AM})}(h_{i3}) \quad (i = 1, 2, \ldots, 5).$$

Step 3. Calculate the deviation sore functions of H_i $(i = 1, 2, \ldots, 5)$. Take the calculation of $\mathbb{DS}^{(D^{(x^2,\mathrm{AM})},\mathrm{WAM})}(H_1)$ for example,

$$\begin{aligned} &\mathbb{DS}^{(D^{(x^2,\mathrm{AM})},\mathrm{WAM})}(H_1) \\ &= 0.3 \times DS^{(x^2,\mathrm{AM})}(\{0.5, 0.4, 0.3\}) \\ &\quad + 0.4 \times DS^{(x^2,\mathrm{AM})}(\{0.5, 0.4, 0.2\}) \\ &\quad + 0.3 \times DS^{(x^2,\mathrm{AM})}(\{0.9, 0.6, 0.4\}) \\ &= 0.3 \times \frac{0.5^2 + 0.4^2 + 0.3^2}{3} + 0.4 \times \frac{0.5^2 + 0.4^2 + 0.2^2}{3} \\ &\quad + 0.3 \times \frac{0.9^2 + 0.6^2 + 0.4^2}{3} \\ &= 0.24. \end{aligned}$$

Similarly, we have

$\mathbb{DS}^{(D^{(x^2,\mathrm{AM})},\mathrm{WAM})}(H_2) = 0.30$, $\mathbb{DS}^{(D^{(x^2,\mathrm{AM})},\mathrm{WAM})}(H_3) = 0.44$,

$\mathbb{DS}^{(D^{(x^2,\mathrm{AM})},\mathrm{WAM})}(H_4) = 0.44$, and $\mathbb{DS}^{(D^{(x^2,\mathrm{AM})},}$ WAM)(H5) = 0.33.

Since $\mathbb{DS}^{(D^{(x^2,\mathrm{AM})},\mathrm{WAM})}(H_3) = \mathbb{DS}^{(D^{(x^2,\mathrm{AM})},\mathrm{WAM})}(H_4)$, we then conduct the computation of the dual ones.

$$\begin{aligned}
&\overline{\mathbb{DS}}^{(D^{(f,\mathrm{AM})},\mathrm{WAM})}(H_3)\\
&= 1 - \mathbb{D}^{(D^{(x^2,\mathrm{AM})},\mathrm{WAM})}(H_3, \mathbf{1})\\
&= 1 - \mathrm{WAM}((1 - \overline{\mathrm{DS}}^{(x^2,\mathrm{AM})}(\mathrm{h}_{31}), (1 - \overline{\mathrm{DS}}^{(x^2,\mathrm{AM})}(\mathrm{h}_{32}),\\
&\quad (1 - \overline{DS}^{(x^2,\mathrm{AM})}(h_{31}))\\
&= 1 - 0.3 \times ((1 - \overline{DS}^{(x^2,\mathrm{AM})}(h_{31}))\\
&\quad + 0.4 \times (1 - \overline{DS}^{(x^2,\mathrm{AM})}(h_{32}))\\
&\quad + 0.3 \times (1 - \overline{DS}^{(x^2,\mathrm{AM})}(h_{33})))\\
&= 1 - 0.3 \times 0.125 + 0.4 \times 0.085 + 0.3 \times 0.26\\
&= 1 - 0.15\\
&= 0.85
\end{aligned}$$

and $\overline{\mathbb{DS}}^{(D^{(x^2,\mathrm{AM})},\mathrm{WAM})}(H_4) = 0.84$. The results from the dual NHFSDSF and PHFSDSF are displayed in Fig. 1. The alternative x_3 and x_4 are indiscernible under the score values by the NHFSDSF $\mathbb{DS}^{(D^{(x^2,\mathrm{AM})},\mathrm{WAM})}$, and there exists a slightly difference between the score values calculated from the PHFSDSF $\overline{\mathbb{DS}}^{(D^{(x^2,\mathrm{AM})},\mathrm{WAM})}$.

Step 4. We rank the alternatives by the dual deviation score values of the corresponding HFSs as

$$x_3 \succ_{\mathbb{DS}} x_4 \succ_{\mathbb{DS}} x_5 \succ_{\mathbb{DS}} x_2 \succ_{\mathbb{DS}} x_1.$$

Step 5. Choose x_3 as the best system analysis engineer.

In the following, we are to take an in-depth analysis of the solution processes by the proposed method, and then to compare the methods in [21, 31] with ours.

Back to the calculation process above, we adopted $f(x) = x^2$, $\Delta_1 = \mathrm{AM}$, and $\Delta = \mathrm{WAW}$ to compute the score values of the hesitant fuzzy evaluation information. The relations of attributions were given in advance and the effect function f was set by decision makers, so the operator f and Δ should be kept stationary. We further choose these common-used operators max, min, AM, OWA, and GM for Δ_1 to calculate the score values of HFSs for ranking the five alternatives. The detailed score values are given in Table 3, and the comparisons of score values calculated by the NHFSDSF and PHFSDSF with different Δ_1 are demonstrated in Figs. 2 and 3. From the two figures,

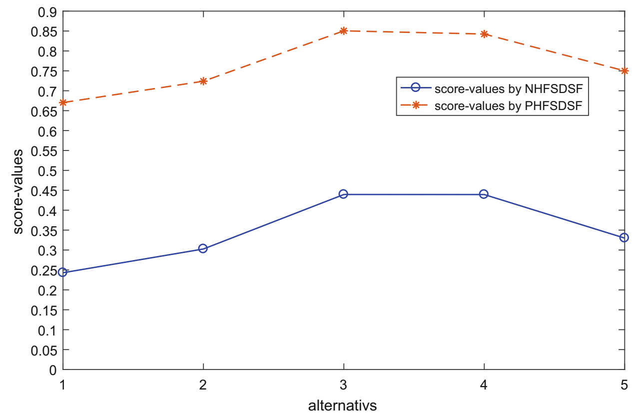

Fig. 1 Comparison of the score values calculated from the dual NHFSDSF and PHFSDSF

the score values of x_3 are better than those of x_4 in most cases, which provides some evidence for choosing x_3 as the best one.

Both the single score value-based method (SSVM) [31] and the entropy TOPSIS-based approach (ENTTA) [21] can be applied to solve the HFMADM problems. We compare our method with them separately.

(1) Comparison with SSVM [31]. The computation process of SSVM is simpler than that of our method, whereas our method seems to be better from the angle of the ranking result. The HFS score functions used in SSVM are exactly the score functions derived from the NHFSDSF. Using the SSVM method with the same inner operator AM and outer operator WAM as ours, one obtains both the score value of x_3 and x_4 are the same as 0.44 (See Table 3). The result from the SSVM method by score function $\mathbb{S}_{\mathrm{WAM}} = \mathbb{DS}^{(D^{(x^2,\mathrm{AM})},\mathrm{WAM})}$ is displayed as

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \succ_{\mathbb{S}_{\mathrm{WAM}}} x_5 \succ_{\mathbb{S}_{\mathrm{WAM}}} x_2 \succ_{\mathbb{S}_{\mathrm{WAM}}} x_1,$$

in which the first two alternatives x_3 and x_4 are indiscernible. The two tied alternatives might trouble the decision makers, and it would take much more efforts of decision makers to solve the problem. By contrast, the score values of x_3 and x_4 by our approach are different with

Table 3 Score values computed from NHFSDSF $\mathbb{DS}s$ and PHFSDSF $\mathbb{DS}$ with different Δ_1

	NHFSDSF					PHFSDSF				
	Max	Min	AM	OWA	GM	Max	Min	AM	OWA	GM
x_1	0.42	0.09	0.24	0.29	0.20	0.48	0.82	0.67	0.71	0.29
x_2	0.55	0.14	0.30	0.36	0.25	0.56	0.93	0.72	0.78	0.21
x_3	0.58	0.30	**0.44**	0.58	0.41	0.78	0.92	**0.85**	0.78	0.12
x_4	0.60	0.27	**0.44**	0.61	0.40	0.76	0.92	**0.84**	0.76	0.11
x_5	0.55	0.12	0.33	0.50	0.26	0.56	0.93	0.75	0.64	0.18

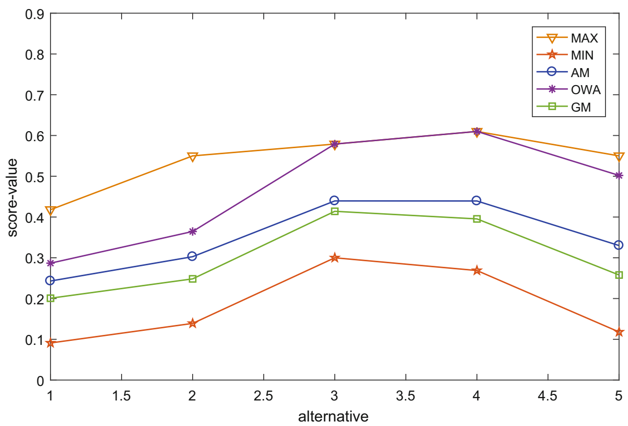

Fig. 2 The NHFSDSF score values calculated from Δ_1 with different operators

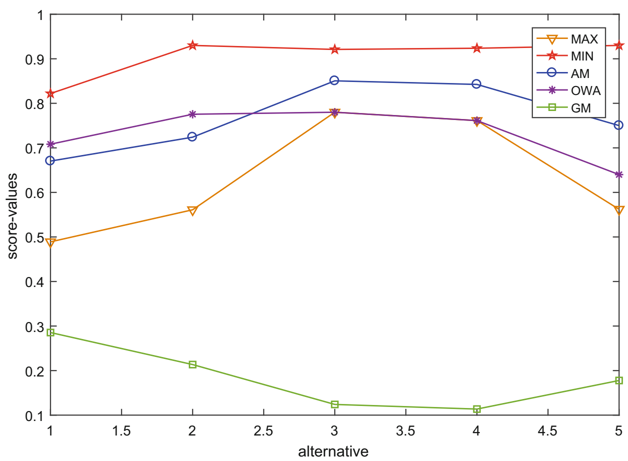

Fig. 3 The PHFSDSF score values calculated from Δ_1 with different operators

0.85 and 0.84 (See Table 3). Moreover, the five score values are all not the same, then the corresponding total ranking helps to select the best alternative immediately.

(2) Comparison with ENTTA [21]. Deriving weights of attributes based on the entropy measure is a notable characteristic of Wei's ENTTA. In our method, the weights of attributes need to be provided subjectively before implementing the algorithm. However, one of the preparatory works of ENTTA is to complement the shorter hesitant fuzzy elements. When facing the problem with more short HFEs, the complement process might result in an unreliable result. Our method makes decisions with the original evaluation information, which could lay the foundation for obtaining a more convincing result.

7 Conclusions

This paper investigated the essence and the generation mechanism of score functions from the perspective of deviation degree. It has been proved that the existing concrete score functions can be easily constructed from the HFE (HFS) deviation score functions. Moreover, the proposed dual deviation score value-based ranking methods have been successfully applied to multi-attribute decision-making problem with hesitant fuzzy information.

From the property analysis and illustrative examples, some conclusions are obtained as follows.

(1) Deviation score functions provide a more general way to construct valuable score functions individually. Then, decision makers could design any particular score functions by setting different parameters according to the requirements of practice.

(2) The ranking method based on the dual deviation score functions performs better than the one on a single score function in discerning different HFEs and HFSs, which leads to a more widespread application of the former in decision making.

(3) When applying some of the deviation score functions in HFMADM problems with hesitant fuzzy elements of different length, the calculation can be deployed without any supplements. These calculations with the original information could lay a foundation for obtaining a more reliable result.

The potential applications of different HFE and HFS score functions in real-world MCDA problems need further exploration. In our future work, we will devote to the theoretical analysis and practical application of score functions in different hesitant fuzzy settings.

(1) The notions of deviation score functions and the corresponding ranking methods could be generalized to the other types of hesitant fuzzy information, such as dual, higher order, interval-valued, and hesitant fuzzy linguistic.

(2) Real-time decision problems are significant and extensively existing in the daily life of human beings and organizations. To explore the potential of deviation scorer functions in solving such issues is another important research topic.

(3) The remarkable performance of hesitant fuzzy sets in clustering analysis and machine learning has attracted much more attention from researches and practitioners. As the important measures in hesitant fuzzy sets, the proposed deviation degree and deviation score functions are expected to be applied in these promising fields.

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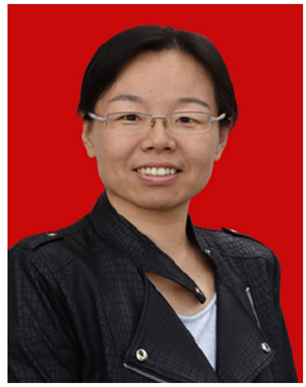

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