The algebraic properties of Concept Lattice

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Abstract

Concept lattice is a powerful tool for data analysis .It has been applied widely in machine learning, knowledge discovery and software engineering and so on. Some aspects of concept lattice have been studied widely such as building lattice and rules extraction , as for its algebraic properties,there has not been discussed systematically .The paper suggests a binary operation between the elements for the set of all concepts in formal context. This turns the concept lattice in general significance into those with operators.we also proved that the concept lattice is a lattice in algebraic significance and studied its algebraic properties. These results provided theoretical foundation and a new method for further study of concept lattice.

Keywords: Concept lattice, Operator, Algebra system

1. Introduction

Concept lattice was proposed by R.wille in 1982[1], which is also known as Galois lattice.Concept lattice is a conceptual hierarchical structure based on binary relation .It is a powerful tool for data analysis . Presently,Concept lattice has been widely used in the field of Web documentary retrieval[2,3,4],digital library [5],software engineering[12,13,14,15],data mining[16,17,18] and knowledge discovery[6,7,8] and so on. The process of building a concept lattice from data set is virtually a process of conceptual clustering. Hasse diagram of concept lattice represents the association between objects and attributes, and reflects the relationship of generalization and specialization among concepts, so it can be regarded as an efficient method for data analysis and knowledge acquisition [9,10,11].However, the concept lattice discussed now are based on the significance of partial order between the concepts of concept sets[1].Concept lattice based on ordered structure have many inconvenience for the discussing of algebraic structure of concept lattice,it also makes some properties such as isomorphism and homomorphism between concept lattice and conceptual categoricalness can not be studied sufficiently.In this paper,we propose operations $\cup$ and $\cap$ between concepts in concept lattice.Thus the concept lattice in general order become algebra system with binary operation $\cup$ and $\cap$.This provide a powerful tool for discovering of algebraic properties of concept lattice and a new method for discussing association between concepts. Along with the further study of the relation between concepts and algebraic properties of concept lattice,the

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mathematical properties of concept lattice will be mined, this will be very helpful for people to understand the essence of concept lattice ,establish foundation for theoretical study of concept lattice and provide new method for further study of concept lattice.

2. Basic concept of concept lattice

The formal context is a triple (U,D,R),where U is a finite set of elements called objects , D is a finite set of elements called attributes.R is a binary relation between U and D.For $x\in U$ and $y\in D$, if the object x has the attribute m,then x and y have the relation R, which we denote by xRy. Now we definite two hypothesis f and g between the power set P(U) of U and the power set P(D) of D as follows:

$$\forall X\in P(U), f(X)=\{y\in D \mid \forall x\in X, xRy\}.$$
$$\forall Y\in P(D), g(D)=\{x\in U \mid \forall y\in Y, xRy\}.$$

Definition 1[1] Let K=(U,D,R) be a formal context, $X\in P(U)$ and $Y\in P(D)$.(X,Y) is called a concept, if f(X)=Y and g(Y)=X hold for X and Y, where X is called the extent of the concept and Y is called the intent of the concept.L(K) denotes the set of all concepts in the formal context.

Definition 2[1] For the formal context K=(U,D,R),let $H_1=(X_1,Y_1)$ and $H_2=(X_2,Y_2)$ be two elements of L(K).If there exists $H_1\le H_2 \Leftrightarrow Y_2\subseteq Y_1$,then "$\le$" is a partial order of L(K),which produce a lattice structure in L(K),called concept lattice of formal context K=(U,D,R),also denoted by L(K).

Table 1 is a formal context, and Figure 1 shows its Hasse diagram.

	a	b	c	d
1	1			
2		1		
3			1	
4				1
5	1	1	1	
6	1	1		1

Table 1

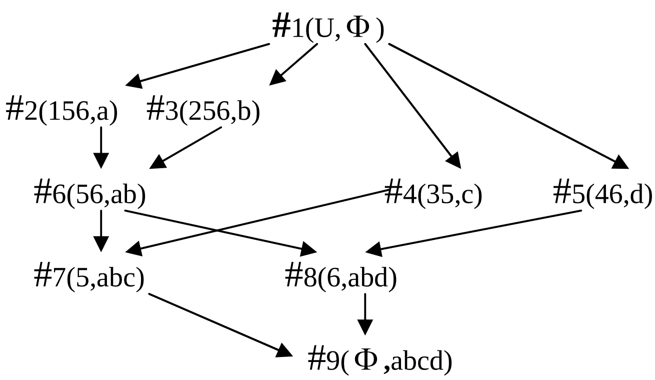

Figure 1

3. Binary operation in concept lattice

For formal context K=(U,D,R),let $H_1=(X_1,Y_1)$ and $H_2=(X_2,Y_2)$ be two elements of L(K).It is not appropriate to define the operation of join and merge for concepts according to $H_1\bigcap H_2=(X_1\bigcap X_2,Y_1\bigcup Y_2)$ and $H_1\bigcup H_2=(X_1\bigcup X_2,Y_1\bigcap Y_2)$.Because the operation of join and merge for concepts according to the above method cannot always get a new concept. For example, in figure 1, for $\#_3(256,b)$ and $\#_5(46,d)$,though we have $(\{2,5,6\}\cap\{4,6\},\{b\}\cup\{d\})=(6,bd)$ and $(\{6\}\cup\{3,5\},\{a,b,d\}\cup\{c\})=(356,\varnothing)$, $(6,bd)$ and $(356,\varnothing)$ are not concepts. Now we propose the reasonable definition of the operation of join and merge for concepts as follows:

Definition 3 Let L(K) be the set of all concepts in formal context K=(U,D,R), $H_1=(X_1,Y_1)$ and $H_2=(X_2,Y_2)$ be two elements of L(K),we define

$$H_1\cup H_2=(g(Y_1\cap Y_2),Y_1\cap Y_2))$$

$$H_1\cap H_2=(X_1\cap X_2,f(X_1\cap X_2))$$

Now we prove that $H_1\cup H_2$ and $H_1\cap H_2$ in the definition are concepts.

Theorem 1. According to the definition 3, $H_1\cup H_2$ and $H_1\cap H_2$ are elements of L(K), i.e., two concepts of K=(U,D,R).

Proof. for every $y\in(Y_1\cup Y_2)$,we have that $y\in Y_1$ or $y\in Y_2$. If $y\in Y_1$, it can easily follows that xRy holds for every $x\in(X_1\cap X_2)\subseteq X_1$. Similarly, if $y\in Y_2$, it can also follows that xRy . So for every $y\in(Y_1\cup Y_2)$, we always have $y\in f(X_1\cap X_2)$. Therefore

$$Y_1\cup Y_2 \subseteq f(X_1\cap X_2). \qquad (3.1)$$

Conversely, for every $x\in X_1\cup X_2$,we have that $x\in X_1$ or $x\in X_2$. Thus,for every $y\in(Y_1\cap Y_2)$,i.e. $y\in Y_1$ and $y\in Y_2$,we have the xRy, hence $x \in g(Y_1\cap Y_2)$. So

$$X_1\cup X_2\subseteq g(Y_1\cap Y_2). \qquad (3.2)$$

To prove $H_1\cap H_2=(X_1\cap X_2,f(X_1\cap X_2))$ be a concept,we only need to prove $g(f(X_1\cap X_2)=X_1\cap X_2$ .Since $g(f(X_1\cap X_2)=\{x\in U \mid \forall y\in f(X_1\cap X_2),xRy\}$,if $x\in g(f(X_1\cap X_2)$ and $Y_1\cup Y_2\subseteq f(X_1\cap X_2)$ (3.1) are satisfied for every $y\in Y_1$ or $y\in Y_2$,then we always have xRy.Since $f(Y_1)=X_1$,it can follow $x\in X_1$. Similarly, we have $x\in X_2$, hence $x\in X_1\cap X_2$. Therefore $g(f(X_1\cap X_2)\subseteq(X_1\cap X_2)$.

Conversely,for $x\in(X_1\cap X_2)$, $\forall y\in f(X_1\cap X_2)$,according to the definition of $f(X_1\cap X_2)$,it can easily follow xRy . Therefore $x\in g(f(X_1\cap X_2)$,we have $g(f(X_1\cap X_2)\supseteq(X_1\cap X_2)$.Thus,we have proven that $g(f(X_1\cap X_2)=X_1\cap X_2$, so $(X_1\cap X_2,f(X_1\cap X_2))$ is a concept, i.e., $(X_1\cap X_2,f(X_1\cap X_2))\in L(K)$.

Now we prove that $H_1\cup H_2=(g(Y_1\cap Y_2),Y_1\cap Y_2)$ is a concept. Similarly, we only need to prove $f(g(Y_1\cap Y_2), Y_1\cap Y_2)=Y_1\cap Y_2$. Let $y\in Y_1\cap Y_2$, for every $x\in g(Y_1\cap Y_2)$,according to the definition of $g(Y_1\cap Y_2)$,we have xRy, i.e.,$y\in f(g(Y_1\cap Y_2)$.Thus $Y_1\cap Y_2\subseteq f(g(Y_1\cap Y_2)$.

On the other hand, for $y\in f(g(Y_1\cap Y_2)$ and for every $x\in g(Y_1\cap Y_2)$,it easily follows xRy. According to (3.2), $X_1\cup X_2\subseteq g(Y_1\cap Y_2)$,so for $x\in X_1$ or $x\in X_2$ and $y\in f(g(Y_1\cap Y_2)$,we always have xRy. Since $f(X_1)=Y_1$,$g(Y_1)=X_1$,$f(X_2)=Y_2$ and $g(Y_2)=X_2$,we can follow that $y\in Y_1\cap Y_2$. So $f(g(Y_1\cap Y_2)\subseteq(Y_1\cap Y_2)$,thus $f(g(Y_1\cap Y_2)=Y_1\cap Y_2$. Therefore $H_1\cup H_2=(g(Y_1\cap Y_2),Y_1\cap Y_2)$ is also a concept.

Corollary 1. The set of concepts L(K) in formal context is closed for operation $\cap$ and $\cup$,thus $(L(K),\cap,\cup)$ forms a algebra system with binary operator $\cap$ and $\cup$.

Theorem 2. The elements $H_1=(X_1,Y_1)$, $H_2=(X_2,Y_2)$ and $H_3=(X_3,Y_3)$ of algebra system $(L(K),\cap,\cup)$ satisfy with the following operation law:

L_1. $H_1\cap H_1=H_1$; $H_1\cup H_1=H_1$; (idempotent law)

L_2. $H_1 \cap H_2 = H_2 \cap H_1$; $H_1 \cup H_2 = H_2 \cup H_1$; (commutative law)

L_3. $(H_1 \cap H_2) \cap H_3 = H_1 \cap (H_2 \cap H_3)$;

$(H_1 \cup H_2) \cup H_3 = H_1 \cup (H_2 \cup H_3)$; (associative law)

L_4. $H_1 \cap (H_1 \cup H_2) = H_1$; $H_1 \cup (H_1 \cap H_2) = H_1$; (absorbing law)

Proof. According to definition ,L_1 and L_2 are apparently true.Now we prove L_3.

Since

$$\begin{aligned}(H_1 \cap H_2) \cap H_3 &= (X_1 \cap X_2, f(X_1 \cap X_2)) \cap (X_3, Y_3)\\ &= ((X_1 \cap X_2) \cap X_3, f(X_1 \cap X_2 \cap X_3))\\ &= (X_1 \cap X_2 \cap X_3, f(X_1 \cap X_2 \cap X_3))\end{aligned}$$

and

$$\begin{aligned}H_1 \cap (H_2 \cap H_3) &= (X_1, Y_1) \cap (X_2 \cap X_3, f(X_2 \cap X_3,))\\ &= (X_1 \cap (X_2 \cap X_3), f(X_1 \cap (X_2 \cap X_3)))\\ &= (X_1 \cap X_2 \cap X_3, f(X_1 \cap X_2 \cap X_3))\end{aligned}$$

Hence $(H_1 \cap H_2) \cap H_3 = H_1 \cap (H_2 \cap H_3)$.

For the second statement of L_3,

$$\begin{aligned}H_1 \cup (H_2 \cup H_3) &= (X_1, Y_1) \cup (g(Y_2 \cap Y_3), Y_2 \cap Y_3)\\ &= (g(Y_1 \cap (Y_2 \cap Y_3)), Y_1 \cap (Y_2 \cap Y_3))\\ &= (g(Y_1 \cap Y_2 \cap Y_3), Y_1 \cap Y_2 \cap Y_3)\end{aligned}$$

and

$$\begin{aligned}(H_1 \cup H_2) \cup H_3 &= (g(Y_1 \cap Y_2), Y_1 \cap Y_2) \cup (X_3, Y_3)\\ &= (g((Y_1 \cap Y_2) \cap Y_3), (Y_1 \cap Y_2) \cap Y_3)\\ &= (g(Y_1 \cap Y_2 \cap Y_3), Y_1 \cap Y_2 \cap Y_3)\end{aligned}$$

Hence $(H_1 \cup H_2) \cup H_3 = H_1 \cup (H_2 \cup H_3)$.

Thus,we have proven L_3.

Now we prove L_4.

$$\begin{aligned}H_1 \cap (H_1 \cup H_2) &= (X_1, Y_1) \cap (g(Y_1 \cap Y_2), Y_1 \cap Y_2)\\ &= (X_1 \cap g(Y_1 \cap Y_2), f(X_1 \cap g(Y_1 \cap Y_2)))\end{aligned}$$

Since $f(X_1)=Y_1$,we have $X_1 \subseteq g(Y_1 \cap Y_2)$. Therefore

$$(X_1 \cap g(Y_1 \cap Y_2), f(X_1 \cap g(Y_1 \cap Y_2))) = (X_1, f(X_1)) = (X_1, Y_1) = H_1$$

Thus,we have $H_1 \cap (H_1 \cup H_2) = H_1$. However

$$\begin{aligned}H_1 \cup (H_1 \cap H_2) &= (X_1, Y_1) \cup (X_1 \cap X_2, f(X_1 \cap X_2))\\ &= (g(Y_1 \cap f(X_1 \cap X_2)), Y_1 \cap f(X_1 \cap X_2))\\ &= (g(Y_1), Y_1) = (X_1, Y_1) = H_1\end{aligned}$$

Thus we have proven L_4.

Theorem 3. Let L(K) be a concept lattice in formal context K=(U,D,R).Let $H_1=(X_1,Y_1)$ and $H_2=(X_2,Y_2)$ be elements of L(K).Then we have *L.u.b.*$\{H_1,H_2\} = H_1 \cup H_2$ (supremum),*G.l.b.*$\{H_1,H_2\} = H_1 \cap H_2$ (infimum).

Proof . Let $H=(X,Y)=$ *L.u.b.*$\{H_1,H_2\}$,then $H_1 \le H$ and $H_2 \le H$, and further $Y \subseteq Y_1$ and $Y \subseteq Y_2$. i.e. $Y \subseteq Y_1 \cap Y_2$,hence $(g(Y_1 \cap Y_2), Y_1 \cap Y_2) \le H$. Since $Y_1 \cap Y_2 \subseteq Y_1$ and $Y_1 \cap Y_2 \subseteq Y_2$,it can follow $H_1 \le (g(Y_1 \cap Y_2), Y_1 \cap Y_2)$ and $H_2 \le (g(Y_1 \cap Y_2), Y_1 \cap Y_2)$. Because H is the supremum, we have $H \le (g(Y_1 \cap Y_2), Y_1 \cap Y_2)$, i.e., $H= H_1 \cup H_2$.Similarly *G.l.b*$\{H_1,H_2\} = H_1 \cap H_2$.

From theorem3, we have theorem as follows:

Theorem 4. In the significance of the definition 3,the concept lattice L(K) in formal context K=(U,D,R) is a lattice in algebraic system significance,which satisfywith all the properties of algebra lattice .

Corollary 2. The concept lattice L(K) in formal context K=(U,D,R) is a complete lattice,any of its subset all have supremum and infimum.

Theorem 5. The concept lattice $(L(K), \bigcap, \bigcup)$ in formal context K=(U,D,R)is a lattice with unit element 1 and zero element 0.

Proof. Let $0=(\Phi,D)$, $1=(U, \Phi)$, $\forall H_1=(X_1,Y_1)\in L(K)$,Since

$$H_1\bigcup (\Phi,D)=(g(Y_1\bigcap D), Y_1\bigcap D)=(g(Y_1), Y_1)=(X_1,Y_1)=H_1$$

and

$$H_1\bigcap(U, \Phi)=(X_1\bigcap U,f(X_1\bigcap U))=(X_1, f(X_1))=(X_1,Y_1)=H_1$$

So 0 and 1 are respectively zero element and unit element of $(L(K), \bigcap,\bigcup)$.

Theorem 6. For two elements $H_1=(X_1,Y_1)$ and $H_2=(X_2,Y_2)$ of concept lattice $(L(K), \bigcap,\bigcup)$,if $f(X_1\bigcap X_2)=Y_1\bigcap Y_2$ and $g(Y_1\bigcap Y_2)=X_1\bigcup X_2$, then $(L(K), \bigcap,\bigcup)$ is a distributive lattice.

Proof. Since

$$(X_1,Y_1) \bigcup (X_2,Y_2)=(X_1\bigcup X_2, Y_1\bigcap Y_2)$$

and $(X_1,Y_1) \bigcap (X_2,Y_2) = (X_1\bigcap X_2, Y_1\bigcup Y_2)$, therefore

$$\begin{aligned} H_1\bigcap (H_2\bigcup H_3) &= (X_1,Y_1) \bigcap (X_2,Y_2) \bigcup (X_3,Y_3) \\ &= (X_1,Y_1) \bigcap (X_2\bigcup X_3, Y_2\bigcap Y_3) \\ &= (X_1\bigcap (X_2\bigcup X_3), Y_1\bigcup (Y_2\bigcap Y_3)) \\ &=((X_1\bigcap X_2) \bigcup (X_1\bigcap X_3), (Y_1\bigcup Y_2)\bigcap (Y_1\bigcup Y_3)) \end{aligned}$$

and

$$\begin{aligned} (H_1\bigcap H_2) \bigcup (H_1\bigcup H_3) &= (X_1\bigcap X_2, Y_1\bigcap Y_2) \bigcup (X_1\bigcap X_3, Y_1\bigcap Y_3) \\ &=((X_1\bigcap X_2) \bigcup (X_1\bigcap X_3,),(Y_1\bigcup Y_2) \bigcap (Y_1\bigcup Y_3)) \end{aligned}$$

i.e.,

$$H_1\bigcap (H_2\bigcup H_3)=(H_1\bigcup H_2) \bigcap (H_1\bigcup H_3)$$

Similarly , we can prove $H_1\bigcup (H_2\bigcap H_3)=(H_1\bigcup H_2) \bigcap (H_1\bigcup H_3)$. Thus $(L(K), \bigcap,\bigcup)$ is a distributive lattice.

Theorem 7. Let L(K) be a concept lattice in formal context K=(U,D,R). Let $H_1=(X_1,Y_1)$ and $H_2=(X_2,Y_2)$ be elements of L(K). The following proposition are equivalent.

a) $H_1\leq H_2$

b) $H_1\bigcap H_2=H_1$; $H_1\bigcup H_2=H_2$.

Proof. Suppose that a) is true . Since $Y_2\subseteq Y_1$ and $X_1\subseteq X_2$,we have

$$(X_1,Y_1)\bigcap (X_2,Y_2)=(X_1\bigcap X_2,f(X_1\bigcap X_2))=(X_1, f(X_1))=(X_1,Y_1)=H_1$$

and

$$H_1\bigcup H_2=(X_1,Y_1) \bigcup (X_2,Y_2)=(g(Y_1\bigcap Y_2) ,Y_1\bigcap Y_2))=(g(Y_2),Y_2)=(X_2,Y_2)=H_2,$$

Hence, b) is true.

Suppose that b) is true. Since $H_1\bigcap H_2=H_1 \Leftrightarrow (X_1,Y_1) =(X_1\bigcap X_2,f(X_1\bigcap X_2))$,from (3.1) ,it follows that $Y_1\bigcup Y_2 \subseteq f(X_1\bigcap X_2) = Y_1$ and $Y_1\bigcup Y_2\subseteq Y_1$ .Thus $Y_2\subseteq Y_1$,i.e.,$H_1\leq H_2$. Hence a) is true.

4. Conclusions

Concept lattice have been getting widely application in many fields with its particular advantage. The paper gives binary operation for concept lattice, which will establish the theory foundation for the studying isomorphism and homomorphism in concept lattice,provide new tool for analyzing the association between concepts in formal context, and new approach for building concept lattice.

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