The algebraic properties of Concept Lattice

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Abstract

Concept lattice is a powerful tool for data analysis. It has been applied widely in machine learning, knowledge discovery and software engineering and so on. Some aspects of concept lattice have been studied widely such as building lattice and rules extraction, as for its algebraic properties, there has not been discussed systematically. The paper suggests a binary operation between the elements for the set of all concepts in formal context. This turns the concept lattice in general significance into those with operators. We also proved that the concept lattice is a lattice in algebraic significance and studied its algebraic properties. These results provided theoretical foundation and a new method for further study of concept lattice.

Keywords: Concept lattice, Operator, Algebra system

1. Introduction

Concept lattice was proposed by R. Wille in 1982[1], which is also known as Galois lattice. Concept lattice is a conceptual hierarchical structure based on binary relation. It is a powerful tool for data analysis. Presently, concept lattice has been widely used in the field of Web documentary retrieval[2,3,4], digital library [5], software engineering[12,13,14,15], data mining[16,17,18] and knowledge discovery[6,7,8] and so on. The process of building a concept lattice from data set is virtually a process of conceptual clustering. Hasse diagram of concept lattice represents the association between objects and attributes, and reflects the relationship of generalization and specialization among concepts, so it can be regarded as an efficient method for data analysis and knowledge acquisition [9,10,11]. However, the concept lattice discussed now are based on the significance of partial order between the concepts of concept sets[1]. Concept lattice based on ordered structure have many inconvenience for the discussing of algebraic structure of concept lattice, it also makes some properties such as isomorphism and homomorphism between concept lattice and conceptual categoricalness can not be studied sufficiently. In this paper, we propose operations ∪ and ∩ between concepts in concept lattice. Thus the concept lattice in general order become algebra system with binary operation ∪ and ∩. This provide a powerful tool for discovering of algebraic properties of concept lattice and a new method for discussing association between concepts. Along with the further study of the relation between concepts and algebraic properties of concept lattice, the

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mathematical properties of concept lattice will be mined, this will be very helpful for people to understand the essence of concept lattice, establish foundation for theoretical study of concept lattice and provide new method for further study of concept lattice.

2. Basic concept of concept lattice

The formal context is a triple (U,D,R), where U is a finite set of elements called objects, D is a finite set of elements called attributes. R is a binary relation between U and D. For x ∈ U and y ∈ D, if the object x has the attribute m, then x and y have the relation R, which we denote by xRy. Now we define two hypothesis f and g between the power set P(U) of U and the power set P(D) of D as follows:

∀ X ∈ P(U), f(X) = {y ∈ D | ∀ x ∈ X, xRy}.
∀ Y ∈ P(D), g(D) = {x ∈ U | ∀ y ∈ Y, xRy}.

Definition 1[1] Let K=(U,D,R) be a formal context, X ∈ P(U) and Y ∈ P(D), (X,Y) is called a concept, if f(X)=Y and g(Y)=X hold for X and Y, where X is called the extent of the concept and Y is called the intent of the concept. L(K) denotes the set of all concepts in the formal context.

Definition 2[1] For the formal context K=(U,D,R), let H1=(X1,Y1) and H2=(X2,Y2) be two elements of L(K). If there exists H1 ≤ H2 ⇔ Y2 ⊆ Y1, then “≤” is a partial order of L(K), which produce a lattice structure in L(K), called concept lattice of formal context K=(U,D,R), also denoted by L(K).

Table 1 is a formal context, and Figure 1 shows its Hasse diagram.

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Table 1

3. Binary operation in concept lattice

For formal context K=(U,D,R), let H1=(X1,Y1) and H2=(X2,Y2) be two elements of L(K). It is not appropriate to define the operation of join and merge for concepts according to H1 ∩ H2=(X1 ∩ X2, Y1 ∪ Y2) and H1 ∪ H2=(X1 ∪ X2, Y1 ∩ Y2). Because the operation of join
and merge for concepts according to the above method cannot always get a new concept. For example, in figure 1, for \( \#_{3}(256,b) \) and \( \#_{4}(46,d) \), though we have \( \{(2,5,6) \cap \{4,6\}, \{b\} \cup \{d\}\}=(6, bd) \) and \( \{(6) \cup \{3,5\}, \{a,b,d\} \cup \{c\}\}=(356, \emptyset) \), \( (6, bd) \) and (356, \emptyset) are not concepts. Now we propose the reasonable definition of the operation of join and merge for concepts as follows:

**Definition 3** Let \( L(K) \) be the set of all concepts in formal context \( K=(U,D,R) \), and merge for concepts according to the above method cannot always get a new concept. For example, in figure 1, for \( \#_{3}(256,b) \) and \( \#_{4}(46,d) \), though we have \( \{(2,5,6) \cap \{4,6\}, \{b\} \cup \{d\}\}=(6, bd) \) and \( \{(6) \cup \{3,5\}, \{a,b,d\} \cup \{c\}\}=(356, \emptyset) \), \( (6, bd) \) and (356, \emptyset) are not concepts. Now we propose the reasonable definition of the operation of join and merge for concepts as follows:

**Theorem 1.** According to the definition 3, \( H_{1}\cup H_{2} \) and \( H_{1}\cap H_{2} \) are elements of \( L(K) \), i.e., two concepts of \( K=(U,D,R) \).

**Proof.** for every \( y \in ( Y_{1} \cup Y_{2} ) \), we have that \( y \in Y_{1} \) or \( y \in Y_{2} \). If \( y \in Y_{1} \), it can easily follows that \( xRy \) holds for every \( x \in ( X_{1} \cap X_{2} ) \subseteq X_{1} \). Similarly, if \( y \in Y_{2} \), it can also follows that \( xRy \). So for every \( y \in ( Y_{1} \cup Y_{2} ) \), we always have \( y \in f(X_{1} \cap X_{2}) \). Therefore \( Y_{1} \cap Y_{2} \subseteq f(X_{1} \cap X_{2}) \). (3.1)

Conversely, for every \( x \in X_{1} \cup X_{2} \), we have that \( x \in X_{1} \) or \( x \in X_{2} \). Thus, for every \( y \in ( Y_{1} \cap Y_{2} ) \), i.e., \( y \in Y_{1} \) and \( y \in Y_{2} \), we have the \( xRy \), hence \( x \in g(Y_{1} \cap Y_{2}) \). So \( X_{1} \cap X_{2} \subseteq g(Y_{1} \cap Y_{2}) \). (3.2)

To prove \( H_{1} \cap H_{2} = ( X_{1} \cap X_{2} ), f(X_{1} \cap X_{2}) \) be a concept, we only need to prove \( g(f(X_{1} \cap X_{2}) = X_{1} \cap X_{2} ) \). Since \( g(f(X_{1} \cap X_{2}) \subseteq \{x \in U \mid \forall y \in f(X_{1} \cap X_{2}), xRy\} \), if \( x \in g(f(X_{1} \cap X_{2}) \) and \( Y_{1} \cap Y_{2} \subseteq f(X_{1} \cap X_{2}) \) (3.1) are satisfied for every \( y \in Y_{1} \) or \( y \in Y_{2} \), then we always have \( xRy \). Since \( f(Y_{1}) = X_{1} \), it can follow \( x \in X_{1} \). Similarly, we have \( x \in X_{2} \), hence \( x \in X_{1} \cap X_{2} \). Therefore \( g(f(X_{1} \cap X_{2}) \subseteq ( X_{1} \cap X_{2} ) \).

Conversely, for \( x \in ( X_{1} \cap X_{2} ) \), \( \forall y \in f(X_{1} \cap X_{2}) \), according to the definition of \( f(X_{1} \cap X_{2}) \), it can easily follow \( xRy \). Therefore \( x \in g(f(X_{1} \cap X_{2}) \), we have \( g(f(X_{1} \cap X_{2}) \subseteq ( X_{1} \cap X_{2} ) \). Thus, we can easily prove that \( g(f(X_{1} \cap X_{2}) = X_{1} \cap X_{2} \) i.e., \( X_{1} \cap X_{2} , f(X_{1} \cap X_{2}) \) is a concept, i.e., \( ( X_{1} \cap X_{2} , f(X_{1} \cap X_{2}) ) \) is a concept.

Now we prove that \( H_{1} \cup H_{2} = ( Y_{1} \cap Y_{2} ) \subseteq ( Y_{1} \cap Y_{2} ) \) is a concept. Similarly, we only need to prove \( f(g(Y_{1} \cap Y_{2}) = Y_{1} \cap Y_{2} ) \). Let \( y \in Y_{1} \cap Y_{2} \), for every \( x \in g(Y_{1} \cap Y_{2}) \), according to the definition of \( g(Y_{1} \cap Y_{2}) \), we always have \( xRy \), i.e., \( y \in f(g(Y_{1} \cap Y_{2}) \). Thus \( Y_{1} \cap Y_{2} \subseteq f(g(Y_{1} \cap Y_{2}) \).

On the other hand, for \( y \in f(g(Y_{1} \cap Y_{2}) \) and every \( x \in g(Y_{1} \cap Y_{2}) \), it easily follows \( xRy \). According to (3.2), \( X_{1} \cap X_{2} \subseteq g(Y_{1} \cap Y_{2}) \), so for \( x \in X_{1} \) or \( x \in X_{2} \) and \( y \in f(g(Y_{1} \cap Y_{2}) \), we always have \( xRy \). Since \( f(X_{1}) = Y_{1} \), \( g(Y_{1}) = X_{1} \), \( f(X_{2}) = Y_{2} \) and \( g(Y_{2}) = X_{2} \), we can follow that \( y \in Y_{1} \cap Y_{2} \). So \( f(g(Y_{1} \cap Y_{2}) \subseteq ( Y_{1} \cap Y_{2} ) \), i.e., \( f(g(Y_{1} \cap Y_{2}) \subseteq Y_{1} \cap Y_{2} \). Therefore \( H_{1} \cup H_{2} = ( Y_{1} \cap Y_{2} ) \subseteq ( Y_{1} \cap Y_{2} ) \) is a concept.

**Corollary 1.** The set of concepts \( L(K) \) in formal context is closed for operation \( \cap \) and \( \cup \), thus \( (L(K), \cap, \cup) \) forms a algebra system with binary operator \( \cap \) and \( \cup \).

**Theorem 2.** The elements \( H_{1}=(X_{1},Y_{1}) \), \( H_{2}=(X_{2},Y_{2}) \) and \( H_{3}=(X_{3},Y_{3}) \) of algebra system \( (L(K), \cap, \cup) \) satisfy the following operation law:

\[
L_{1}: \quad H_{1} \cap H_{2} = H_{1}; \quad H_{1} \cup H_{1} = H_{1}; \quad (\text{idempotent law})
\]
L.  \( H_1 \cap H_2 = H_2 \cap H_1; H_1 \cup H_2 = H_2 \cup H_1; \) (commutative law)
L.  \( (H_1 \cap H_2) \cap H_3 = H_2 \cap (H_2 \cap H_3); \)
\( (H_1 \cup H_2) \cup H_3 = H_2 \cup (H_2 \cup H_3); \) (associative law)
L.  \( H_1 \cap (H_1 \cup H_2) = H_1; H_1 \cup (H_1 \cap H_2) = H_1; \) (absorbing law)
Proof. According to definition, L. \( 1 \) and L. \( 2 \) are apparently true. Now we prove L. \( 3 \).
Since
\[
(H_1 \cap H_2) \cap H_3 = (X_1 \cap X_2, f(X_1 \cap X_2)) \cap (X_3, Y_3) = (X_1 \cap X_2 \cap X_3, f(X_1 \cap X_2 \cap X_3))
\]
and
\[
H_1 \cap (H_2 \cap H_3) = (X_1, Y_1) \cap (X_2 \cap X_3, f(X_2 \cap X_3)) = (X_1 \cap X_2 \cap X_3, f(X_1 \cap X_2 \cap X_3))
\]
Hence \( (H_1 \cap H_2) \cap H_3 = H_1 \cap (H_2 \cap H_3) \).
For the second statement of L. \( 3 \),
\[
H_1 \cup (H_2 \cup H_3) = (X_1, Y_1) \cup (g(Y_2 \cap Y_3), Y_2 \cap Y_3) = (g(Y_1 \cap (Y_2 \cap Y_3)), Y_1 \cap (Y_2 \cap Y_3)) = (g(Y_1 \cap Y_2 \cap Y_3), Y_1 \cap Y_2 \cap Y_3)
\]
and
\[
(H_1 \cup H_2) \cup H_3 = (g(Y_1 \cap Y_2), Y_1 \cap Y_2) \cup (X_3, Y_3) = (g((Y_1 \cap Y_2) \cap Y_3), (Y_1 \cap Y_2) \cap Y_3) = (g(Y_1 \cap Y_2 \cap Y_3), Y_1 \cap Y_2 \cap Y_3)
\]
Hence \( (H_1 \cup H_2) \cup H_3 = H_1 \cup (H_2 \cup H_3) \).
Thus, we have proven L. \( 3 \).
Now we prove L. \( 4 \).
\[
H_1 \cap (H_1 \cup H_2) = (X_1, Y_1) \cap (g(Y_1 \cap Y_2), Y_1 \cap Y_2) = (X_1 \cap g(Y_1 \cap Y_2), f(X_1 \cap g(Y_1 \cap Y_2)))
\]
Since \( f(X_1) = Y_1 \), we have \( X_1 \subseteq g(Y_1 \cap Y_2) \). Therefore
\[
(X_1 \cap g(Y_1 \cap Y_2), f(X_1 \cap g(Y_1 \cap Y_2))) = (X_1, f(X_1)) = (X_1, Y_1) = H_1
\]
Thus, we have \( H_1 \cap (H_1 \cup H_2) = H_1 \). However
\[
H_1 \cup (H_1 \cap H_2) = (X_1, Y_1) \cup (X_1 \cap X_2, f(X_1 \cap X_2)) = (g(Y_1 \cap f(X_1 \cap X_2)), Y_1 \cap f(X_1 \cap X_2)) = (g(Y_1), Y_1) = (X_1, Y_1) = H_1
\]
Thus we have proven L. \( 4 \).

Theorem 3. Let L(\( K \)) be a concept lattice in formal context \( K=(U,D,R) \). Let \( H_1=(X_1, Y_1) \) and \( H_2=(X_2, Y_2) \) be elements of \( L(\( K \)) \). Then we have \( L.u.b. \{H_1, H_2\} = H_1 \cup H_2 \) (supremum), \( G.l.b. \{H_1, H_2\} = H_1 \cap H_2 \) (infimum).
Proof. Let \( H=(X, Y)=L.u.b. \{H_1, H_2\} \), then \( H \subseteq H_1 \) and \( H \subseteq H_2 \), and further \( Y \subseteq Y_1 \) and \( Y \subseteq Y_2 \), i.e., \( Y \subseteq Y_1 \cap Y_2 \), hence \( g(Y_1 \cap Y_2), Y_1 \cap Y_2 \) \( \leq \) \( H \). Since \( Y_1 \cap Y_2 \subseteq Y_1 \) and \( Y_1 \cap Y_2 \subseteq Y_2 \), it can follow \( H \subseteq (g(Y_1 \cap Y_2), Y_1 \cap Y_2) \) and \( H \subseteq (g(Y_1 \cap Y_2), Y_1 \cap Y_2) \). Because \( H \) is the supremum, we have \( H \leq (g(Y_1 \cap Y_2), Y_1 \cap Y_2) \), i.e., \( H = H_1 \cup H_2 \). Similarly \( G.l.b. \{H_1, H_2\} = H_1 \cap H_2 \).
From theorem 3, we have theorem as follows:

**Theorem 4.** In the significance of the definition 3, the concept lattice $L(K)$ in formal context $K=(U,D,R)$ is a lattice in algebraic system significance, which satisfies with all the properties of algebra lattice.

**Corollary 2.** The concept lattice $L(K)$ in formal context $K=(U,D,R)$ is a complete lattice, any of its subset all have supremum and infimum.

**Theorem 5.** The concept lattice $(L(K), \cap, \cup)$ in formal context $K=(U,D,R)$ is a lattice with unit element 1 and zero element 0.

Proof. Let $0=(\Phi,D), 1=(U, \Phi), \forall H_1= (X_1, Y_1) \in L(K)$, Since

$$H_1 \cup (\Phi,D)=(g(Y_1 \cap D), Y_1 \cap D)=(g(Y_1), Y_1)=(X_1, Y_1)=H_1$$

and

$$H_1 \cap (U, \Phi)=(X_1 \cap U, f(X_1 \cap U))=(X_1, f(X_1))=(X_1, Y_1)=H_1$$

So 0 and 1 are respectively zero element and unit element of $(L(K), \cap, \cup)$.

**Theorem 6.** For two elements $H_1=(X_1, Y_1)$ and $H_2=(X_2, Y_2)$ of concept lattice $(L(K), \cap, \cup)$, if $f(X_1 \cap X_2)= Y_1 \cap Y_2$ and $g(Y_1 \cap Y_2)= X_1 \cup X_2$, then $(L(K), \cap, \cup)$ is a distributive lattice.

Proof. Since

$$(X_1, Y_1) \cup (X_2, Y_2)=(X_1 \cup X_2, Y_1 \cap Y_2)$$

and

$$(X_1, Y_1) \cap (X_2, Y_2)=(X_1 \cap X_2, Y_1 \cup Y_2), \text{ therefore }$$

$$H_1 \cap (H_2 \cup H_3)=(X_1, Y_1) \cap (X_2, Y_2) \cup (X_3, Y_3)$$

$$=((X_1 \cap X_2) \cup (X_1 \cap X_2), (Y_1 \cup Y_2) \cap (Y_1 \cup Y_3))$$

and

$$(H_1 \cap H_2) \cap (H_1 \cup H_3)=(X_1 \cap X_2, Y_1 \cap Y_2) \cup (X_1 \cap X_2, Y_1 \cap Y_3)$$

$$=((X_1 \cap X_2) \cup (X_1 \cap X_2), (Y_1 \cup Y_2) \cap (Y_1 \cup Y_3))$$

i.e.,

$$H_1 \cap (H_2 \cup H_3)=(H_1 \cup H_2) \cap (H_1 \cup H_3)$$

Similarly, we can prove $H_1 \cup (H_2 \cap H_3)=(H_1 \cup H_2) \cap (H_1 \cap H_3)$. Thus $(L(K), \cap, \cup)$ is a distributive lattice.

**Theorem 7.** Let $L(K)$ be a concept lattice in formal context $K=(U,D,R)$. Let $H_1=(X_1, Y_1)$ and $H_2=(X_2, Y_2)$ be elements of $L(K)$. The following proposition are equivalent.

a) $H_1 \leq H_2$

b) $H_1 \cap H_2= H_1; H_1 \cup H_2= H_2$.

Proof. Suppose that a) is true. Since $Y_2 \subseteq Y_1$ and $X_1 \subseteq X_2$, we have

$$(X_1, Y_1) \cap (X_2, Y_2)=(X_1 \cap X_2, f(X_1 \cap X_2))=(X_1, f(X_1))=(X_1, Y_1)=H_1$$

and

$$H_1 \cup H_2=(X_1, Y_1) \cup (X_2, Y_2)=(g(Y_1 \cap Y_2), Y_1 \cap Y_2)=(g(Y_2), Y_2)=(X_2, Y_2)=H_2$$

Hence, b) is true.

Suppose that b) is true. Since $H_1 \cap H_2= H_1 \iff (X_1, Y_1)= (X_1 \cap X_2, f(X_1 \cap X_2))$, from (3.1), it follows that $Y_1 \cup Y_2 \subseteq f(X_1 \cap X_2)= Y_1$ and $Y_1 \cup Y_2 \subseteq Y_1$. Thus $Y_2 \subseteq Y_1$, i.e., $H_1 \leq H_2$. Hence a) is true.

4. Conclusions
Concept lattice have been getting widely application in many fields with its particular advantage. The paper gives binary operation for concept lattice, which will establish the theory foundation for the studying isomorphism and homomorphism in concept lattice, provide new tool for analyzing the association between concepts in formal context, and new approach for building concept lattice.

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