Computers and Mathematics with Applications xx (xxxx) xxx–xxx

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Interval ordered information systems

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ARTICLE INFO

Article history:
Received 1 November 2007
Received in revised form 28 March 2008
Accepted 17 April 2008

Keywords:
Interval information systems
Interval decision tables
Dominance relation
Rough sets
Attribute reduction

ABSTRACT

Interval information systems are generalized models of single-valued information systems. By introducing a dominance relation to interval information systems, we propose a ranking approach for all objects based on dominance classes and establish a dominance-based rough set approach, which is mainly based on substitution of the indiscernibility relation by the dominance relation. Furthermore, we discuss interval ordered decision tables and dominance rules. To simplify knowledge representation and extract much simpler dominance rules, we propose attribute reductions of interval ordered information systems and decision tables that eliminate only the information that are not essential from the viewpoint of the ordering of objects or dominance rules. The approaches show how to simplify an interval ordered information system and find dominance rules directly from an interval ordered decision table. These results will be helpful for decision-making analysis in interval information systems.

1. Introduction

Rough set theory, introduced by Pawlak [1,2], has been conceived as a tool to conceptualize and analyze various types of data. It can be used in the attribute-value representation model to describe the dependencies among attributes and evaluate the significance of attributes and derive decision rules. It has important applications to intelligence decision and cognitive sciences, as a tool for dealing with vagueness and uncertainty of facts, and in classification [3–10]. Rough-set-based data analysis starts from a data table, called information systems. The information systems contains data about objects of interest, characterized by a finite set of attributes [11–17]. It is often interesting to discover some dependency relationships (patterns) among attributes.

The original rough set theory does not consider attributes with preference-ordered domains, that is, criteria. However, in many real situations, we are often faced with the problems where the ordering of properties of the considered attributes plays a crucial role. One such problem is the ordering of objects. For this reason, Greco, Matarazzo, and Slowinski [18–21] proposed an extension of rough set theory, called the dominance-based rough set approach (DRSA) to take into account the ordering properties of criteria. This innovation is mainly based on the substitution of the indiscernibility relation by a dominance relation. In DRSA, where condition attributes are criteria and classes are preference ordered, the knowledge approximated is a collection of upward and downward unions of classes and the dominance classes are sets of objects defined by using a dominance relation. In recent years, many studies have been made in DRSA [22–25].

Interval information systems are an important type of data tables, and generalized models of single-valued information systems. In recent years, some problems of decision making in the context of interval information systems have been

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0898-1221/$ – see front matter © 2008 Published by Elsevier Ltd
doi:10.1016/j.camwa.2008.04.021

Table 1
An interval information system

U	a_1	a_2	a_3	a_4	a_5
x_1	1	[0, 1]	2	1	[1, 2]
x_2	[0, 1]	0	[1, 2]	0	1
x_3	[0, 1]	0	[1, 2]	1	1
x_4	0	0	1	0	1
x_5	2	[1, 2]	3	[1, 2]	[2, 3]
x_6	[0, 2]	[1, 2]	[1, 3]	[1, 2]	[2, 3]
x_7	1	1	2	1	2
x_8	[1, 2]	[1, 2]	[2, 3]	2	[2, 3]
x_9	[1, 2]	2	[2, 3]	[0, 2]	3
x_{10}	2	2	3	[0, 1]	3

studied [26–31]. Most of them are based on the concept of a possible degree between two interval numbers [27,30,31]. So far, however, how to make a decision by a dominance relation has not been reported in interval information systems. In this paper, we aim to introduce a dominance relation to interval information systems and interval decision tables, and to establish a rough set approach based on this dominance relation for decision-making analysis in the context of interval numbers.

The rest of this paper is organized as follows. By reviewing some preliminary concepts, a dominance relation is introduced to interval information systems and some of its important properties are investigated in Section 2. In Section 3, through the notions of dominance degree and whole dominance degree, a ranking approach with dominance-class nature is established, which is based on possible place of each object in the rank. In Section 4, a dominance-based rough set approach is established in interval ordered information systems, which is mainly based on the substitution of the indiscernibility relation by a dominance relation. In Section 5, to simplify knowledge representation, a criterion-reduction approach to an interval ordered information system is proposed, which eliminates only those criterions not essential from the viewpoint of the ordering of objects. In Section 6, the notion of interval ordered decision tables is given and dominance rules extracted from this type of decision tables are also discussed. In Section 7, an approach to relative attribute reductions is presented in interval ordered decision tables for extracting much simpler dominance rules. In Section 8, through a venture-investment issue, it is illustrated that how to make a decision by using the approaches proposed in this paper. Finally, we conclude this paper with a summary in Section 9.

2. Dominance relation in interval information systems

In this section, we introduce a dominance relation to an interval information system, and obtain some of its important properties.

An *interval information system* (IIS) is a quadruple $S = (U, AT, V, f)$, where U is a finite non-empty set of objects and AT is a finite non-empty set of attributes, $V = \bigcup_{a\in AT} V_a$ and V_a is a domain of attribute a, $f : U \times AT \to V$ is a total function such that $f(x, a) \in V_a$ for every $a \in AT, x \in U$, called an information function, where V_a is a set of interval numbers. Denoted by

$$f(x, a) = [a^L(x), a^U(x)] = \{p \mid a^L(x) \leqslant p \leqslant a^U(x), a^L(x), a^U(x) \in \mathbf{R}\},$$

we call it the interval number of x under the attribute a. In particular, $f(x, a)$ would degenerate into a real number if $a^L(x) = a^U(x)$. Under this consideration, we regard a single-valued information system as a special form of interval information systems.

Example 2.1. An interval information system is presented in Table 1, where $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$, $AT = \{a_1, a_2, a_3, a_4, a_5\}$.

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In practical decision-making analysis, we always consider a binary dominance relation between objects that are possibly dominant in terms of values of an attributes set in an interval information system. In general, an increasing preference and a decreasing preference are considered by a decision maker. If the domain of an attribute is ordered according to a decreasing or increasing preference, then the attribute is a criterion.

Definition 2.1. An interval information system is called an *interval ordered information system* (IOIS) if all attributes are criterions.

It is assumed that the domain of a criterion $a \in AT$ is completely pre-ordered by an outranking relation $\succcurlyeq_a$; $x \succcurlyeq_a y$ means that x is at least as good as (outranks) y with respect to the criterion a. For a subset of attributes $A \subseteq AT$, we define $x \succcurlyeq_A y \Leftrightarrow \forall a \in A, x \succcurlyeq_a y$. In other words, x is at least as good as y with respect to all attributes in A.

In the following, we introduce a dominance relation that identifies dominance classes to an interval ordered information system. In a given IOIS, we say that x dominates y with respect to $A \subseteq AT$ if $x \succcurlyeq_A y$, and denoted by $xR_A^{\geqslant} y$. That is

$$R_A^{\geqslant} = \{(y, x) \in U \times U \mid y \succcurlyeq_A x\}.$$

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Obviously, if $(y, x) \in R_A^{\geqslant}$, then y dominates x with respect to A. In other words, y may have a better property than x with respect to A in reality.

Analogously, the relation $R_A^{\leqslant}$ (called a dominated relation) can be defined as follows:

$$R_A^{\leqslant} = \{(y, x) \in U \times U \mid x \succcurlyeq_A y\}.$$

Given $A \subseteq AT$ and $A = A_1 \cup A_2$, where the attributes set A_1 according to increasing preference and A_2 according to decreasing preference. Let us define these two binary relations more precisely as follows:

$$\begin{aligned} R_A^{\geqslant} &= \{(y, x) \in U \times U \mid a_1^L(y) \geqslant a_1^L(x),\ a_1^U(y) \geqslant a_1^U(x) (\forall a_1 \in A_1); a_2^L(y) \leqslant a_2^L(x),\ a_2^U(y) \leqslant a_2^U(x) (\forall a_2 \in A_2)\} \\ &= \{(y, x) \in U \times U \mid (y, x) \in R_A^{\geqslant}\}, \end{aligned}$$

$$\begin{aligned} R_A^{\leqslant} &= \{(y, x) \in U \times U \mid a_1^L(y) \leqslant a_1^L(x),\ a_1^U(y) \leqslant a_1^U(x) (\forall a_1 \in A_1); a_2^L(y) \geqslant a_2^L(x),\ a_2^U(y) \geqslant a_2^U(x) (\forall a_2 \in A_2)\} \\ &= \{(y, x) \in U \times U \mid (y, x) \in R_A^{\leqslant}\}. \end{aligned}$$

From the definition of $R_A^{\geqslant}$ and $R_A^{\leqslant}$, the following properties can be easily obtained.

Property 2.1. *Let $S = (U, AT, V, f)$ be an interval ordered information system and $A \subseteq AT$, then*

$$R_A^{\geqslant} = \bigcap_{a \in A} R_{\{a\}}^{\geqslant}, \qquad R_A^{\leqslant} = \bigcap_{a \in A} R_{\{a\}}^{\leqslant}.$$

Property 2.2. *Let $S = (U, AT, V, f)$ be an interval ordered information system and $A \subseteq AT$. Then*

(1) $R_A^{\geqslant}$, $R_A^{\leqslant}$ *are reflexive,*
(2) $R_A^{\geqslant}$, $R_A^{\leqslant}$ *are unsymmetric and*
(3) $R_A^{\geqslant}$, $R_A^{\leqslant}$ *are transitive.*

The dominance class induced by the dominance relation $R_A^{\geqslant}$ is the set of objects dominating x, i.e.,

$$\begin{aligned} [x]_A^{\geqslant} &= \{y \in U \mid a_1^L(y) \geqslant a_1^L(x),\ a_1^U(y) \geqslant a_1^U(x) (\forall a_1 \in A_1); a_2^L(y) \leqslant a_2^L(x),\ a_2^U(y) \leqslant a_2^U(x) (\forall a_2 \in A_2)\} \\ &= \{y \in U \mid (y, x) \in R_A^{\geqslant}\} \end{aligned}$$

and the set of objects dominated by x,

$$\begin{aligned} [x]_A^{\leqslant} &= \{y \in U \mid a_1^L(y) \leqslant a_1^L(x),\ a_1^U(y) \leqslant a_1^U(x) (\forall a_1 \in A_1); a_2^L(y) \geqslant a_2^L(x),\ a_2^U(y) \geqslant a_2^U(x) (\forall a_2 \in A_2)\} \\ &= \{y \in U \mid (x, y) \in R_A^{\geqslant}\}, \end{aligned}$$

where $[x]_A^{\geqslant}$ describes the set of objects that may dominates x and $[x]_A^{\leqslant}$ describes the set of objects that may be dominated by x in terms of A in an interval ordered information system, which are called the A-dominating set and the A-dominated set with respect to $x \in U$, respectively.

Remark. In many practical issues, one also can define the dominance relation on the universe with interval values through using other ways. For example, there are the following definitions:

(1) $R_A^{\geqslant 1} = \{(y, x) \in U \times U \mid a_1^L(y) \geqslant a_1^U(x) (\forall a_1 \in A_1); a_2^L(y) \leqslant a_2^U(x) (\forall a_2 \in A_2)\}$,
(2) $R_A^{\geqslant 2} = \{(y, x) \in U \times U \mid a_1^L(y) \geqslant a_1^L(x) (\forall a_1 \in A_1); a_2^L(y) \leqslant a_2^L(x) (\forall a_2 \in A_2)\}$ and
(3) $R_A^{\geqslant 3} = \{(y, x) \in U \times U \mid a_1^U(y) \geqslant a_1^U(x) (\forall a_1 \in A_1); a_2^U(y) \leqslant a_2^U(x) (\forall a_2 \in A_2)\}$.

Each of these three dominance relations is based on a particular practical meaning, which can induce the dominance class of each object on the universe accordingly. In this investigation, we do not deal with the relationship among these dominance relations, but rough set approach, ranking problem and attribute reduction in interval ordered information systems. Furthermore, through using each of these dominance relations, one can obtain similar research results as the rest part of this paper. Therefore, for convenience, we only adopt the dominance relation $R_A^{\geqslant}$ for studying interval ordered information systems in this paper.

For simplicity, without any loss of generality, in the following we only consider attributes with increasing preference.

Property 2.3. *Let $S = (U, AT, V, f)$ be an interval ordered information system and $A, B \subseteq AT$, we have that*

(1) *if $B \subseteq A \subseteq AT$, then $R_B^{\geqslant} \supseteq R_A^{\geqslant} \supseteq R_{AT}^{\geqslant}$;*
(2) *if $B \subseteq A \subseteq AT$, then $[x]_B^{\geqslant} \supseteq [x]_A^{\geqslant} \supseteq [x]_{AT}^{\geqslant}$;*
(3) *if $x_j \in [x_i]_A^{\geqslant}$, then $[x_j]_A^{\geqslant} \subseteq [x_i]_A^{\geqslant}$ and $[x_i]_A^{\geqslant} = \bigcup\{[x_j]_A^{\geqslant} : x_j \in [x_i]_A^{\geqslant}\}$; and*
(4) $[x_i]_A^{\geqslant} = [x_j]_A^{\geqslant}$ *iff $f(x_i, a) = f(x_j, a)$ $(\forall a \in A)$.*

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Proof. Let $B \subseteq A \subseteq AT$, (1) and (2) are straightforward.

(3) If $x_j \in [x_i]_A^{\wedge\geqslant}$, it follows from the dominance relation $R_A^{\geqslant}$ that the interval number $f(x_j, a)$ is bigger than the interval number $f(x_i, a)$ for arbitrary $a \in A$, i.e., $a^L(x_j) \geqslant a^L(x_i)$ and $a^U(x_j) \geqslant a^U(x_i)$. Analogously, for $\forall x \in [x_j]_A^{\geqslant}$, we have that $a^L(x) \geqslant a^L(x_j)$ and $a^U(x) \geqslant a^U(x_j)$ for arbitrary $a \in A$. Therefore, $a^L(x) \geqslant a^L(x_i)$ and $a^U(x) \geqslant a^U(x_i)$ $(\forall a \in A)$. Thus we have $x \in [x_i]_A^{\geqslant}$, i.e., $[x_j]_A^{\geqslant} \subseteq [x_i]_A^{\geqslant}$. Therefore $[x_i]_A^{\geqslant} = \bigcup\{[x_j]_A^{\geqslant} : x_j \in [x_i]_A^{\geqslant}\}$.

(4) "⇒" When $[x_i]_A^{\geqslant} = [x_j]_A^{\geqslant}$, it follows from (3) that $[x_j]_A^{\geqslant} \subseteq [x_i]_A^{\geqslant}$, i.e., $a^L(x_j) \geqslant a^L(x_i)$ and $a^U(x_j) \geqslant a^U(x_i)$ for any $a \in A$. Analogously, we have that $a^L(x_i) \geqslant a^L(x_j)$ and $a^U(x_i) \geqslant a^U(x_j)$ for any $a \in A$. Thus, $a^L(x_i) = a^L(x_j)$ and $a^U(x_i) = a^U(x_j)$ for every $a \in A$, i.e., $f(x_i, a) = f(x_j, a)$ $(\forall a \in A)$.

"⇐" If $f(x_i, a) = f(x_j, a)$ $(\forall a \in A)$, then $a^L(x_i) = a^L(x_j)$ and $a^U(x_i) = a^U(x_j)$ for any $a \in A$. Therefore, from the definition of the set of objects dominating x, it is easy to get $[x_i]_A^{\geqslant} = [x_j]_A^{\geqslant}$.

This completes the proof. □

Let $U/R_A^{\geqslant}$ denote classification on the universe, which is the family set $\{[x]_A^{\geqslant} \mid x \in U\}$. Any element from $U/R_A^{\geqslant}$ will be called a dominance class with respect to A. Dominance classes in $U/R_A^{\geqslant}$ do not constitute a partition of U in general. They constitute a covering of U.

These notions and properties mentioned above can be understood through the following example.

Example 2.2 (*Continued From* Example 2.1). Compute the classification induced by the dominance relation $R_A^{\geqslant}$ in Table 1.

From Table 1, one can get that

$$U/R_{AT}^{\geqslant} = \{[x_1]_{AT}^{\geqslant}, [x_2]_{AT}^{\geqslant}, \ldots, [x_{10}]_{AT}^{\geqslant}\},$$

where

$$\begin{aligned}
&[x_1]_{AT}^{\geqslant} = \{x_1, x_5, x_7, x_8\}, \qquad [x_2]_{AT}^{\geqslant} = \{x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}\},\\
&[x_3]_{AT}^{\geqslant} = \{x_1, x_3, x_5, x_6, x_7, x_8\}, \qquad [x_4]_{AT}^{\geqslant} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\},\\
&[x_5]_{AT}^{\geqslant} = \{x_5\}, \qquad [x_6]_{AT}^{\geqslant} = \{x_5, x_6, x_8\}, \qquad [x_7]_{AT}^{\geqslant} = \{x_5, x_7, x_8\},\\
&[x_8]_{AT}^{\geqslant} = \{x_8\}, \qquad [x_9]_{AT}^{\geqslant} = \{x_9\}, \qquad [x_{10}]_{AT}^{\geqslant} = \{x_{10}\}.
\end{aligned}$$

And

$$U/R_{AT}^{\leqslant} = \{[x_1]_{AT}^{\leqslant}, [x_2]_{AT}^{\leqslant}, \ldots, [x_{10}]_{AT}^{\leqslant}\},$$

where

$$\begin{aligned}
&[x_1]_{AT}^{\leqslant} = \{x_1, x_2, x_3, x_4\}, \qquad [x_2]_{AT}^{\leqslant} = \{x_2, x_4\}, \qquad [x_3]_{AT}^{\leqslant} = \{x_2, x_3, x_4\},\\
&[x_4]_{AT}^{\leqslant} = \{x_4\}, \qquad [x_5]_{AT}^{\leqslant} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}, \qquad [x_6]_{AT}^{\leqslant} = \{x_2, x_3, x_4, x_6\},\\
&[x_7]_{AT}^{\leqslant} = \{x_1, x_2, x_3, x_4, x_7\}, \qquad [x_8]_{AT}^{\leqslant} = \{x_1, x_2, x_3, x_4, x_6, x_7, x_8\},\\
&[x_9]_{AT}^{\leqslant} = \{x_2, x_4, x_9\}, \qquad [x_{10}]_{AT}^{\leqslant} = \{x_2, x_4, x_{10}\}.
\end{aligned}$$

From this example, we easily verify (3) of Property 3.3. Obviously, dominance classes in $U/R_{AT}^{\geqslant}$ constitute a covering of U.

In practical applications, we can define various dominance relations according to various semantic interpretations and obtain corresponding dominance class of each object with respect to some dominance relation.

3. Ranking for all objects in IOIS

There are two classes of problems in intelligent decision-making: one is to find satisfactory results through ranking with information aggregation, and the other is to find dominance rules through relations. In this section, we only focus on how to rank all objects by using the dominance relation $R_A^{\geqslant}$ in an interval ordered information system.

In [32], Qiu et al. defined a concept of dominance degree for ranking all objects in classical ordered information systems. In the following, we introduce a dominance degree between two objects and a whole dominance degree of a object in order to decide the place of each object in final rank in an interval ordered information system.

Definition 3.1. Let $S = (U, AT, V, f)$ be an IOIS and $A \subseteq AT$. *Dominance degree* between two objects with respect to the dominance relation $R_A^{\geqslant}$ is defined as

$$D_A(x_i, x_j) = \frac{|\sim [x_i]_A^{\geqslant} \cup [x_j]_A^{\geqslant}|}{|U|},$$

where $|\cdot|$ denotes the cardinality of a set, $x_i,\ x_j \in U$.

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From this definition, one can obtain the following property.

Property 3.1. *$D_A(x_i, x_j)$ has the following properties*

(1) $\frac{1}{|U|} \leq D_A(x_i, x_j) \leq 1$;
(2) *if* $(x_j, x_k) \in R_A^{\geqslant}$, *then* $D_A(x_i, x_j) \leqslant D_A(x_i, x_k)$; *and*
(3) *if* $(x_j, x_k) \in R_A^{\geqslant}$, *then* $D_A(x_j, x_i) \geqslant D_A(x_k, x_i)$.

Proof. (1) is straightforward.

(2) It easily follows from Property 2.2 that the dominance relation $R_A^{\geqslant}$ is transitive. Hence, if $(x_j, x_k) \in R_A^{\geqslant}$, we have that $[x_j]_A^{\geqslant} \subseteq [x_k]_A^{\geqslant}$. Therefore

$$\begin{aligned} D_A(x_i, x_j) - D_A(x_i, x_k) &= \frac{1}{|U|}(| \sim [x_i]_A^{\geqslant} \cup [x_j]_A^{\geqslant}| - | \sim [x_i]_A^{\geqslant} \cup [x_k]_A^{\geqslant}|) \\ &\leqslant \frac{1}{|U|}(| \sim [x_i]_A^{\geqslant} \cup [x_k]_A^{\geqslant}| - | \sim [x_i]_A^{\geqslant} \cup [x_k]_A^{\geqslant}|) \\ &= 0, \end{aligned}$$

that is $D_A(x_i, x_j) \leqslant D_A(x_i, x_k)$.

(3) Similar to (2), we have that $[x_j]_A^{\geqslant} \subseteq [x_k]_A^{\geqslant}$, hence $\sim [x_j]_A^{\geqslant} \supseteq \sim [x_k]_A^{\geqslant}$. Thus

$$\begin{aligned} D_A(x_j, x_i) - D_A(x_k, x_i) &= \frac{1}{|U|}(| \sim [x_j]_A^{\geqslant} \cup [x_i]_A^{\geqslant}| - | \sim [x_k]_A^{\geqslant} \cup [x_i]_A^{\geqslant}|) \\ &\geqslant \frac{1}{|U|}(| \sim [x_k]_A^{\geqslant} \cup [x_i]_A^{\geqslant}| - | \sim [x_k]_A^{\geqslant} \cup [x_i]_A^{\geqslant}|) \\ &= 0, \end{aligned}$$

that is $D_A(x_j, x_i) \geqslant D_A(x_k, x_i)$.

This completes the proof. □

In fact, a dominance degree between two objects with respect to the dominance relation $R_A^{\geqslant}$ also can be defined by the rough membership function, which is as follows

$$D_A^*(x_i, x_j) = \frac{|[x_i]_A^{\geqslant} \cap [x_j]_A^{\geqslant}|}{|[x_i]_A^{\geqslant}|}.$$

However, this definition has some limitations for characterizing dominance degree between any two objects in interval ordered information systems. For example, Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ and $AT = \{a\}$, where $a(x_i) = i$. For the definition of the dominance class, one can know that $[x_1]_{\{a\}}^{\geqslant} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, $[x_5]_{\{a\}}^{\geqslant} = \{x_5, x_6, x_7, x_8\}$, $[x_7]_{\{a\}}^{\geqslant} = \{x_7, x_8\}$ and $[x_8]_{\{a\}}^{\geqslant} = \{x_8\}$. Hence, we have that

$$D_{\{a\}}^*(x_1, x_5) = \frac{|[x_1]_{\{a\}}^{\geqslant} \cap [x_5]_{\{a\}}^{\geqslant}|}{|[x_1]_{\{a\}}^{\geqslant}|} = \frac{4}{8} = \frac{1}{2}, \qquad D_{\{a\}}^*(x_7, x_8) = \frac{|[x_7]_{\{a\}}^{\geqslant} \cap [x_8]_{\{a\}}^{\geqslant}|}{|[x_7]_{\{a\}}^{\geqslant}|} = \frac{1}{2}.$$

That is $D_{\{a\}}^*(x_1, x_5) = D_{\{a\}}^*(x_7, x_8)$. In other words, the above definition can not be used to well characterize the dominance relation between these two objects with respect to $R_{\{a\}}^{\geqslant}$ in this situation. If we adopt the dominance degree in Definition 3.1, then one can obtain that

$$D_{\{a\}}(x_1, x_5) = \frac{| \sim [x_1]_{\{a\}}^{\geqslant} \cup [x_5]_{\{a\}}^{\geqslant}|}{|U|} = \frac{4}{8} = \frac{1}{2}, \qquad D_{\{a\}}(x_7, x_8) = \frac{| \sim [x_7]_{\{a\}}^{\geqslant} \cup [x_8]_{\{a\}}^{\geqslant}|}{|U|} = \frac{7}{8},$$

i.e., $D_{\{a\}}(x_1, x_5) < D_{\{a\}}(x_7, x_8)$. This can depict the practical means of the example. Therefore, one can draw a conclusion: the dominance degree $D_A(x_i, x_j)$ has better performance than the term $D_A^*(x_i, x_j)$ for characterizing the dominance relation between any two objects in interval ordered information systems.

From Definition 3.1, let $(x_i, x_j) \in U \times U$, we can construct a *dominance relation matrix* with respect to A induced by the dominance relation $R_A^{\geqslant}$. From this matrix, the *whole dominance degree* of each object can be calculated according to the following formula

$$D_A(x_i) = \frac{1}{|U| - 1} \sum_{j \neq i} D_A(x_i, x_j), \quad x_i, x_j \in U.$$

As a result of the above discussions, we come to the following two corollaries.

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Corollary 3.1. *Let $S = (U, AT, V, f)$ be an IOIS and $A \subseteq AT$. If $R_A^{\geqslant} = R_{AT}^{\geqslant}$, then $D_A(x_i, x_j) = D_{AT}(x_i, x_j)$, $x_i, x_j \in U$.*

Corollary 3.2. *Let $S = (U, AT, V, f)$ be an IOIS and $A \subseteq AT$. If $R_A^{\geqslant} = R_{AT}^{\geqslant}$, then $D_A(x_i) = D_{AT}(x_i)$ for any $x_i \in U$.*

Similar to the definition of dominance degree, we can give a concept called dominated degree between two objects in an interval ordered information system as follows

$$D'_A(x_i, x_j) = \frac{|\sim [x_i]_A^{\leqslant} \cup [x_j]_A^{\leqslant}|}{|U|}.$$

Let $(x_i, x_j) \in U \times U$, we thus can construct a *dominanced relation matrix* with respect to A induced by the relation $R_A^{\leqslant}$. From this matrix, the *whole dominanced degree* of each object can also be calculated according to the following formula

$$D'_A(x_i) = \frac{1}{|U| - 1} \sum_{j \neq i} D'_A(x_i, x_j), \quad x_i,\ x_j \in U.$$

Through these two concepts, we can also design a ranking approach for all objects.

From the whole dominance degree of each object on the universe, we can rank all objects according to the number of $D_A(x_i)$. A larger number implies a better object. This idea can be understood by the following example.

Example 3.1 (*Continued From Example 2.2*). Rank all objects in U according to the dominance relation $R_{AT}^{\geqslant}$.

From the definition of dominance degree, we can get the dominance relation matrix as

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 0.7 & 0.8 & 0.9 & 0.7 & 0.6 & 0.6 \\
0.5 & 1 & 0.7 & 1 & 0.2 & 0.4 & 0.4 & 0.2 & 0.2 & 0.2 \\
0.8 & 1 & 1 & 1 & 0.5 & 0.7 & 0.7 & 0.5 & 0.4 & 0.4 \\
0.4 & 0.9 & 0.6 & 1 & 0.1 & 0.3 & 0.3 & 0.1 & 0.1 & 0.1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0.9 & 0.9 & 0.9 \\
0.9 & 1 & 1 & 1 & 0.8 & 1 & 0.9 & 0.8 & 0.7 & 0.7 \\
1 & 1 & 1 & 1 & 0.8 & 0.9 & 1 & 0.9 & 0.7 & 0.7 \\
1 & 1 & 1 & 1 & 0.9 & 1 & 1 & 1 & 0.9 & 0.9 \\
0.9 & 1 & 0.9 & 1 & 0.9 & 0.9 & 0.9 & 0.9 & 1 & 0.9 \\
0.9 & 1 & 0.9 & 1 & 0.9 & 0.9 & 0.9 & 0.9 & 0.9 & 1
\end{pmatrix}.$$

Therefore, one can obtain that

$$\begin{aligned}
&D_{AT}(x_1) = 0.81, \quad D_{AT}(x_2) = 0.42, \quad D_{AT}(x_3) = 0.67, \quad D_{AT}(x_4) = 0.32, \\
&D_{AT}(x_5) = 0.97, \quad D_{AT}(x_6) = 0.87, \quad D_{AT}(x_7) = 0.89, \quad D_{AT}(x_8) = 0.97, \\
&D_{AT}(x_9) = 0.92, \quad D_{AT}(x_{10}) = 0.92.
\end{aligned}$$

In the following, we rank all objects according to the number of $D_{AT}(x_i)$. A object with larger number implies a better object.

$$\begin{pmatrix} x_5 \\ x_8 \end{pmatrix} \succcurlyeq \begin{pmatrix} x_9 \\ x_{10} \end{pmatrix} \succcurlyeq x_7 \succcurlyeq x_6 \succcurlyeq x_1 \succcurlyeq x_3 \succcurlyeq x_2 \succcurlyeq x_4.$$

Note that one can obtain a corresponding rank for all objects by using any dominance relation with a practical semantic interpretation.

4. Rough set approach to IOIS

The original rough set approach proved to be very useful in dealing with inconsistency problems following from the information granulation. The original rough set idea is failing, however, when preference-orders of attributes domains (criterion) are to be taken into account [19]. In this section, we investigate the problem of set approximation with respect to a dominance relation $R_A^{\geqslant}$ in interval ordered information systems.

Definition 4.1. Let $S = (U, AT, V, f)$ be an IOIS. For any $X \subseteq U$ and $A \subseteq AT$, the lower and upper approximations of X with respect to the dominance relation $R_A^{\geqslant}$ are defined as follows:

$$\underline{R_A^{\geqslant}}(X) = \{x \in U \mid [x]_A^{\geqslant} \subseteq X\}$$

and

$$\overline{R_A^{\geqslant}}(X) = \{x \in U \mid [x]_A^{\geqslant} \cap X \neq \varnothing\}.$$

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From Definition 4.1, one can easily notice that $\underline{R_A^{\geqslant}}(X)$ is a set of objects that belong to X with certainty and $\overline{R_A^{\geqslant}}(X)$ is a set of objects that possibly belong to X. $Bn_A(X) = \overline{R_A^{\geqslant}}(X) - \underline{R_A^{\geqslant}}(X)$ denotes a boundary of the rough set.

Moreover, from Definition 4.1, one can easily obtain the following properties.

Property 4.1. *Let $S = (U, AT, V, f)$ be an IOIS, $X \subseteq U$, $A \subseteq AT$ and $R_A^{\geqslant}$ a dominance relation, then*

(1) $\underline{R_A^{\geqslant}}(\emptyset) = \overline{R_A^{\geqslant}}(\emptyset) = \emptyset$, $\underline{R_A^{\geqslant}}(U) = \overline{R_A^{\geqslant}}(U) = U$;
(2) $\underline{R_A^{\geqslant}}(X) \subseteq X \subseteq \overline{R_A^{\geqslant}}(X)$;
(3) $\underline{R_A^{\geqslant}}(\underline{R_A^{\geqslant}}(X)) = \underline{R_A^{\geqslant}}(X)$, $\overline{R_A^{\geqslant}}(\overline{R_A^{\geqslant}}(X)) = \overline{R_A^{\geqslant}}(X)$;
(4) $\underline{R_A^{\geqslant}}(X) = \sim \overline{R_A^{\geqslant}}(\sim X)$, $\overline{R_A^{\geqslant}}(X) = \sim \underline{R_A^{\geqslant}}(\sim X)$; *and*
(5) $\underline{R_A^{\geqslant}}(X) \subseteq \underline{R_{AT}^{\geqslant}}(X)$, $\overline{R_A^{\geqslant}}(X) \supseteq \overline{R_{AT}^{\geqslant}}(X)$, *and* $Bn_{AT}(X) \subseteq Bn_A(X)$.

Property 4.2. *Let $S = (U, AT, V, f)$ be an IOIS, $X, Y \subseteq U$, $A \subseteq AT$ and $R_A^{\geqslant}$ a dominance relation, then*

(1) *if* $X \subseteq Y$, *then* $\underline{R_A^{\geqslant}}(X) \subseteq \underline{R_A^{\geqslant}}(Y)$, $\overline{R_A^{\geqslant}}(X) \subseteq \overline{R_A^{\geqslant}}(Y)$;
(2) $\underline{R_A^{\geqslant}}(X \cap Y) = \underline{R_A^{\geqslant}}(X) \cap \underline{R_A^{\geqslant}}(Y)$;
(3) $\overline{R_A^{\geqslant}}(X \cup Y) = \overline{R_A^{\geqslant}}(X) \cup \overline{R_A^{\geqslant}}(Y)$;
(4) $\overline{R_A^{\geqslant}}(X \cap Y) \subseteq \overline{R_A^{\geqslant}}(X) \cap \overline{R_A^{\geqslant}}(Y)$; *and*
(5) $\underline{R_A^{\geqslant}}(X \cup Y) \supseteq \underline{R_A^{\geqslant}}(X) \cup \underline{R_A^{\geqslant}}(Y)$.

The lower and upper approximations of X with respect to the dominance relation $R_A^{\geqslant}$ can be used to extract dominance rules by a decision maker, where one can extract dominance rules with certainty by using $\underline{R_A^{\geqslant}}(X)$ and can extract possible dominance rules by using $Bn_A(X) = \overline{R_A^{\geqslant}}(X) - \underline{R_A^{\geqslant}}(X)$.

Property 4.3. *Let $S = (U, AT, V, f)$ be an IOIS and $A \subseteq AT$. If $R_A^{\geqslant} = R_{AT}^{\geqslant}$, then $\underline{R_A^{\geqslant}}(X) = \underline{R_{AT}^{\geqslant}}(X)$ and $\overline{R_A^{\geqslant}}(X) = \overline{R_{AT}^{\geqslant}}(X)$.*

Uncertainty of a rough set is due to the existence of a borderline region. The greater the borderline region of a rough set, the lower is the accuracy of the rough set. In order to measure the imprecision of a rough set induced by dominance relation $R_A^{\geqslant}(X)$ in an interval ordered information system, we introduce a concept of accuracy measure in the following.

Definition 4.2. Let $S = (U, AT, V, f)$ be an IOIS, $X \subseteq U$ and $A \subseteq AT$. Accuracy measure of X with respect to the dominance relation $R_A^{\geqslant}$ is defined as

$$\alpha(R_A^{\geqslant}, X) = \frac{|\underline{R_A^{\geqslant}}(X)|}{|\overline{R_A^{\geqslant}}(X)|} = \frac{|\underline{R_A^{\geqslant}}(X)|}{|U| - |\underline{R_A^{\geqslant}}(\sim X)|}.$$

The accuracy measure expresses the degree of completeness of the knowledge about X, given the granularity of $U/R_A^{\geqslant}$. It is easy to see that this measure not only depends on the lower approximation of X but also depends on the lower approximation of $\sim X$.

As a direct result of Property 4.3 and Definition 4.2, we come to the following corollary.

Corollary 4.1. *Let $S = (U, AT, V, f)$ be an IOIS and $A \subseteq AT$. If $R_A^{\geqslant} = R_{AT}^{\geqslant}$, then $\alpha(R_A^{\geqslant}, X) = \alpha(R_{AT}^{\geqslant}, X)$.*

Property 4.4. *Let $S = (U, AT, V, f)$ be an IOIS, $X \subseteq U$ and $B \subseteq A \subseteq AT$, then*

$$\alpha(R_{AT}^{\geqslant}, X) \geqslant \alpha(R_A^{\geqslant}, X) \geqslant \alpha(R_B^{\geqslant}, X).$$

Proof. Since $A \subseteq AT$, it follows from (5) of Property 4.1 that $\underline{R_A^{\geqslant}}(X) \subseteq \underline{R_{AT}^{\geqslant}}(X)$ and $\overline{R_A^{\geqslant}}(X) \supseteq \overline{R_{AT}^{\geqslant}}(X)$. Thus

$$\alpha(R_A^{\geqslant}, X) = \frac{|\underline{R_A^{\geqslant}}(X)|}{|\overline{R_A^{\geqslant}}(X)|} \leqslant \frac{|\underline{R_{AT}^{\geqslant}}(X)|}{|\overline{R_{AT}^{\geqslant}}(X)|} = \alpha(R_{AT}^{\geqslant}, X),$$

that is $\alpha(R_{AT}^{\geqslant}, X) \geqslant \alpha(R_A^{\geqslant}, X)$. Similarly, one can obtain $\alpha(R_A^{\geqslant}, X) \geqslant \alpha(R_B^{\geqslant}, X)$. This completes the proof. □

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Example 4.1. Consider the interval ordered information system in Table 1. Let $A = \{a_1, a_2, a_3, a_5\} \subset AT$ and $X = \{x_1, x_5, x_6, x_8\}$, compute the rough sets of X approximated by $U/R^{\geqslant}_{AT}$ and $U/R^{\geqslant}_{A}$, respectively.

According to Definition 4.1 and Example 2.2, the rough set $(\underline{R^{\geqslant}_{AT}}(X), \overline{R^{\geqslant}_{AT}}(X))$ can be obtained as follows:

$$\underline{R^{\geqslant}_{AT}}(X) = \{x_5, x_6, x_8\}, \qquad \overline{R^{\geqslant}_{AT}}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}.$$

Then we compute the classification induced by the dominance relation $U/R^{\geqslant}_{A}$. From Table 1, it follows that

$$U/R^{\geqslant}_{A} = \{[x_1]^{\geqslant}_{A}, [x_2]^{\geqslant}_{A}, \ldots, [x_{10}]^{\geqslant}_{A}\},$$

where

$$[x_1]^{\geqslant}_{A} = \{x_1, x_5, x_7, x_8, x_9, x_{10}\}, \qquad [x_2]^{\geqslant}_{A} = \{x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}\},$$
$$[x_3]^{\geqslant}_{A} = \{x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}\}, \qquad [x_4]^{\geqslant}_{A} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\},$$
$$[x_5]^{\geqslant}_{A} = \{x_5, x_{10}\}, \qquad [x_6]^{\geqslant}_{A} = \{x_5, x_6, x_8, x_9, x_{10}\}, \qquad [x_7]^{\geqslant}_{A} = \{x_5, x_7, x_8, x_9, x_{10}\},$$
$$[x_8]^{\geqslant}_{A} = \{x_5, x_8, x_9, x_{10}\}, \qquad [x_9]^{\geqslant}_{A} = \{x_9, x_{10}\}, \qquad [x_{10}]^{\geqslant}_{A} = \{x_{10}\}.$$

Similarly, we can calculate the rough set $(\underline{R^{\geqslant}_{A}}(X), \overline{R^{\geqslant}_{A}}(X))$ as follows:

$$\underline{R^{\geqslant}_{A}}(X) = \varnothing, \qquad \overline{R^{\geqslant}_{A}}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}.$$

Therefore, we have that

$$\alpha(R^{\geqslant}_{A}, X) = \frac{|\underline{R^{\geqslant}_{A}}(X)|}{|\overline{R^{\geqslant}_{A}}(X)|} = 0, \qquad \alpha(R^{\geqslant}_{AT}, X) = \frac{|\underline{R^{\geqslant}_{AT}}(X)|}{|\overline{R^{\geqslant}_{AT}}(X)|} = \frac{3}{8}.$$

Thus, $\alpha(R^{\geqslant}_{AT}, X) \geqslant \alpha(R^{\geqslant}_{A}, X)$.

5. Attribute reduction in IOIS

A reduct is a subset of attributes that are jointly sufficient and individually necessary for preserving a particular property of a given information system [33]. The original concept of attribute reduction was first proposed by Pawlak [1,2]. Slezak [34,35] investigated how to obtain approximation reducts from an information system or a decision table. Zhang [32] et al. examined a distribution reduct and a maximum distribution reduct in decision tables. Zhao et al. [33] presented a general definition of an attribute reduct, which brings new insight into the problem of reduct construction.

In the context of dominance relations, to simplify knowledge representation it is necessary to reduce some dispensable attributes in a given interval ordered information system. In this section, an approach to attribute reduction in interval ordered information systems will be established and an illustrative example is employed to show its validity.

In the first part of this section, we investigate an attribute reduction approach to an interval ordered information system.

Definition 5.1. Let $S = (U, AT, V, f)$ be an IOIS and $A \subseteq AT$. If $R^{\geqslant}_{A} = R^{\geqslant}_{AT}$ and $R^{\geqslant}_{B} \neq R^{\geqslant}_{AT}$ for any $B \subset A$, then we call A an *attribute reduction* of S.

Property 5.1. *Let $S = (U, AT, V, f)$ be an IOIS and $A \subseteq AT$. If A is an attribute reduction, then $D_A(x_i, x_j) = D_{AT}(x_i, x_j), x_i, x_j \in U$.*

It is obvious that an attribute reduction of an IOIS is a minimal attribute subset satisfying $R^{\geqslant}_{A} = R^{\geqslant}_{AT}$. An attribute $a \in AT$ is called dispensable with respect to $R^{\geqslant}_{AT}$ if $R^{\geqslant}_{AT} = R^{\geqslant}_{(AT-\{a\})}$; otherwise a is called indispensable. The set of all indispensable attributes is called a *core* with respect to the dominance relation $R^{\geqslant}_{AT}$ and is denoted by $\mathrm{core}(AT)$. An attribute in the core must be in every attribute reduction (like the case in complete/incomplete OIS, an IOIS may have many reductions, denoted by $\mathrm{red}(AT)$). Thus $\mathrm{core}(AT) = \bigcap \mathrm{red}(AT)$. The core may be an empty set.

Let $S = (U, AT, V, f)$ be an IOIS and $A \subseteq AT$. For convenient representation, denoted by

$$\mathrm{Dis}(x, y) = \{a \in A \mid (x, y) \notin R^{\geqslant}_{a}\},$$

then we call $\mathrm{Dis}(x, y)$ a *discernibility attribute set* between x and y, and

$$\mathbf{Dis} = (\mathrm{Dis}(x, y) : x, y \in U)$$

a *discernibility matrix* of an IOIS. Clearly, for $\forall x, y \in U$ we have that $\mathrm{Dis}(x, x) = \varnothing$ and $\mathrm{Dis}(x, y) \cap \mathrm{Dis}(y, x) = \varnothing$.

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Table 2
The discernibility matrix of Table 1

x_i/x_j	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$a_1a_2a_3$	$a_1\cdots a_5$	a_2a_5	$a_1\cdots a_5$	$a_1\cdots a_5$	$a_1a_2a_3a_5$
x_2	$a_1\cdots a_5$	a_4	$\varnothing$	$a_1\cdots a_5$	$a_1\cdots a_5$	$a_1\cdots a_5$	$a_1\cdots a_5$	$a_1\cdots a_5$	$a_1\cdots a_5$	$a_1\cdots a_5$
x_3	$a_1a_2a_3a_5$	$\varnothing$	$\varnothing$	$\varnothing$	$a_1\cdots a_5$	$a_1\cdots a_5$	$a_1a_2a_3a_5$	$a_1\cdots a_5$	$a_1\cdots a_5$	$a_1a_2a_3a_5$
x_4	$a_1\cdots a_5$	a_1a_3	$a_1a_3a_4$	$\varnothing$	$a_1\cdots a_5$	$a_1\cdots a_5$	$a_1\cdots a_5$	$a_1\cdots a_5$	$a_1\cdots a_5$	$a_1\cdots a_5$
x_5	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	a_4	a_2a_5	a_2a_5
x_6	a_1a_3	$\varnothing$	$\varnothing$	$\varnothing$	a_1a_3	$\varnothing$	a_1a_3	$a_1a_3a_4$	$a_1a_2a_3a_5$	$a_1a_2a_3a_5$
x_7	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$a_1\cdots a_5$	$a_1\cdots a_5$	$\varnothing$	$a_1\cdots a_5$	$a_1\cdots a_5$	$a_1a_2a_3a_5$
x_8	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	a_1a_3	$\varnothing$	$\varnothing$	$\varnothing$	a_2a_5	$a_1a_2a_3a_5$
x_9	a_4	$\varnothing$	a_4	$\varnothing$	$a_1a_3a_4$	a_4	a_4	a_4	$\varnothing$	a_1a_3
x_{10}	a_4	$\varnothing$	a_4	$\varnothing$	a_4	a_4	a_4	a_4	a_4	$\varnothing$

The following property provides a judgement method of an attribute reduction of an IOIS.

Property 5.2. *Let $S = (U, AT, V, f)$ be an IOIS, $A \subseteq AT$ and $\mathrm{Dis}(x, y)$ the discernibility attributes set of S with respect to $R^{\geqslant}_{AT}$. Then, $R^{\geqslant}_{AT} = R^{\geqslant}_{A}$ iff $A \cap \mathrm{Dis}(x, y) \neq \varnothing$ ($\mathrm{Dis}(x, y) \neq \varnothing$).*

Proof. "$\Rightarrow$" Let $R^{\geqslant}_{AT} = R^{\geqslant}_{A}$, from the definition of the dominance relation, one can know that $[x]^{\geqslant}_{AT} = [x]^{\geqslant}_{A}$ for any $x \in U$. If some $y \notin [x]^{\geqslant}_{AT}$, then $y \notin [x]^{\geqslant}_{A}$. Therefore, there exists $a \in A$ such that $(x, y) \notin [x]^{\geqslant}_{\{a\}}$. So one has $a \in \mathrm{Dis}(x, y)$. Thus, when $\mathrm{Dis}(x, y) \neq \varnothing$ we have $A \cap \mathrm{Dis}(x, y) \neq \varnothing$.

"$\Leftarrow$" From the definition of the discernibility attribute set, one can know that if $(x, y) \notin [x]^{\geqslant}_{AT}$ for any $(x, y) \in U \times U$, then $\mathrm{Dis}(x, y) \neq \varnothing$. And since $A \cap \mathrm{Dis}(x, y) \neq \varnothing$, there exists $a \in A$ such that $a \in \mathrm{Dis}(x, y)$, i.e., $(x, y) \notin [x]^{\geqslant}_{\{a\}}$. So $(x, y) \notin [x]^{\geqslant}_{A}$. Thus $R^{\geqslant}_{AT} \supseteq R^{\geqslant}_{A}$. On the other hand, it follows from $A \subseteq AT$ that $R^{\geqslant}_{AT} \subseteq R^{\geqslant}_{A}$. Therefore, one has $R^{\geqslant}_{AT} = R^{\geqslant}_{A}$.

This completes the proof. □

Definition 5.2. Let $S = (U, AT, V, f)$ be an IOIS, $A \subseteq AT$ and $\mathrm{Dis}(x, y)$ the discernibility attributes set of S with respect to $R^{\geqslant}_{AT}$. Denoted by

$$M = \bigwedge \left\{ \bigvee \{a : a \in \mathrm{Dis}(x, y)\} : x,\ y \in U \right\},$$

then M is referred to as a discernibility function.

Based on the discernibility function, we can design a practical approach to attribute reductions in an IOIS as follows.

Property 5.3. *Let $S = (U, AT, V, f)$ be an IOIS. The minimal disjunctive normal form of discernibility function M is*

$$M = \bigvee_{k=1}^{t} \left(\bigwedge_{s=1}^{q_k} a_{i_s} \right).$$

Denoted by $B_k = \{a_{i_s} : s = 1, 2, \ldots, q_k\}$, then $\{B_k : k = 1, 2, \ldots, t\}$ are the set of all attribute reductions of this system.

Proof. It follows directly from Property 5.2 and the definition of minimal disjunctive normal form of the discernibility function. □

Property 5.3 provides a practical approach to attribute reduction in an interval ordered information system.

As follows, through employing an illustrative example, we analyze how to obtain attribute reductions from all attributes of an IOIS.

Example 5.1 (*Continued From* Example 3.1). Compute all attribute reductions in Table 2.

From the definition of discernibility matrix, we can obtain the discernibility matrix of this system (see Table 2).

Thus, one can obtain that

$$\begin{aligned} M &= (a_1 \vee a_2 \vee a_3) \wedge (a_1 \vee a_2 \vee a_3 \vee a_4 \vee a_5) \wedge (a_2 \vee a_5) \wedge (a_1 \vee a_2 \vee a_3 \vee a_5) \wedge a_4 \wedge (a_1 \vee a_3) \wedge (a_1 \vee a_3 \vee a_4) \\ &= (a_2 \vee a_5) \wedge (a_1 \vee a_3) \wedge a_4 \\ &= (a_1 \wedge a_2 \wedge a_4) \vee (a_1 \wedge a_4 \wedge a_5) \vee (a_2 \wedge a_3 \wedge a_4) \vee (a_3 \wedge a_4 \wedge a_5). \end{aligned}$$

Therefore, there are four attribute reductions for this interval ordered information system, which are $\{a_1, a_2, a_4\}$, $\{a_1, a_4, a_5\}$, $\{a_2, a_3, a_4\}$ and $\{a_3, a_4, a_5\}$. Since the attribute a_4 being in every attribute reduction, a_4 is the core of this system with respect to the dominance relation $R^{\geqslant}_{AT}$. In other words, this attribute (criterion) is indispensable from the point of view of the ordering of objects in Table 1.

Please cite this article in press as: Y. Qian, et al., Interval ordered information systems, Computers and Mathematics with Applications (2008), doi:10.1016/j.camwa.2008.04.021

6. Interval ordered decision tables and dominance rules

Extracted dominance rules from information systems is another important class of problems in decision-making analysis. In this section, we investigate interval ordered decision tables and dominance rules extracted from this type of decision tables.

An *interval ordered decision table* (IODT) is an interval ordered information system $S = (U, C \cup d, V, f)$, where d ($d \notin C$ and $f(x, d)(x \in U)$ is single-valued) is an overall preference called the decision and all the elements of C are criterions. Furthermore, assume that the decision attribute d makes a partition of U into a finite number of classes; let $\mathbf{D} = \{D_1, D_2, \ldots, D_r\}$ be a set of these classes that are ordered, that is, for all $i, j \leq r$ if $i \geqslant j$, then the objects from D_i are preferred to the objects from D_j.

The sets to be approximated are an upward union and a downward union of classes, which are defined as follows

$$D_i^{\geqslant} = \bigcup_{j \geqslant i} D_j, \qquad D_i^{\leqslant} = \bigcup_{j \leqslant i} D_j, \quad (i \leqslant r).$$

The statement $x \in D_i^{\geqslant}$ means "x belongs to at least class D_i", whereas $x \in D_i^{\leqslant}$ means "x belongs to at most class D_i".

Analogous to the idea of decision approximation in [25], in the following, we give the definitions of the lower and upper approximations of $D_i^{\geqslant}$ $(i \leqslant r)$ with respect to the dominance relation $R_A^{\geqslant}$ in an IODT.

Definition 6.1. Let $S = (U, C \cup d, V, f)$ be an IODT, $A \subseteq C$ and $\mathbf{D} = \{D_1, D_2, \ldots, D_r\}$ is the decision induced by d. Lower and upper approximations of $D_i^{\geqslant}$ $(i \leqslant r)$ with respect to the dominance relation $R_A^{\geqslant}$ are defined as

$$\underline{R_A^{\geqslant}}(D_i^{\geqslant}) = \{x \in U \mid [x]_A^{\geqslant} \subseteq D_i^{\geqslant}\}$$

and

$$\overline{R_A^{\geqslant}}(D_i^{\geqslant}) = \bigcup_{x \in D_i^{\geqslant}} [x]_A^{\geqslant}.$$

Similarly, we define the lower and upper approximations of $D_i^{\leqslant}$ $(i \leqslant r)$ with respect to the dominance relation $R_A^{\geqslant}$ in an IODT.

Definition 6.2. Let $S = (U, C \cup d, V, f)$ be an IODT, $A \subseteq C$ and $\mathbf{D} = \{D_1, D_2, \ldots, D_r\}$ is the decision induced by d. Lower and upper approximations of $D_i^{\leqslant}$ $(i \leqslant r)$ with respect to the dominance relation $R_A^{\geqslant}$ are defined as

$$\underline{R_A^{\geqslant}}(D_i^{\leqslant}) = \{x \in U \mid [x]_A^{\leqslant} \subseteq D_i^{\leqslant}\}$$

and

$$\overline{R_A^{\geqslant}}(D_i^{\leqslant}) = \bigcup_{x \in D_i^{\leqslant}} [x]_A^{\leqslant}.$$

Naturally, the A-boundaries of $D_i^{\geqslant}$ $(i \leqslant r)$ and $D_i^{\leqslant}$ $(i \leqslant r)$ can be defined as

$$Bn_A(D_i^{\geqslant}) = \overline{R_A^{\geqslant}}(D_i^{\geqslant}) - \underline{R_A^{\geqslant}}(D_i^{\geqslant})$$

and

$$Bn_A(D_i^{\leqslant}) = \overline{R_A^{\geqslant}}(D_i^{\leqslant}) - \underline{R_A^{\geqslant}}(D_i^{\leqslant}).$$

The lower approximations $\underline{R_A^{\geqslant}}(D_i^{\geqslant})$ and $\underline{R_A^{\geqslant}}(D_i^{\leqslant})$ can be used to extract certain dominance rules, and the boundaries $Bn_A(D_i^{\geqslant})$ and $Bn_A(D_i^{\leqslant})$ can be used to mine possible dominance rules from an interval ordered decision table.

In [24], an atomic expression over a single attribute a is defined as either $(a, \geqslant)$ (according to increasing preference) or $(a, \leqslant)$ (according to decreasing preference) in an ordered information system. For any $A \subseteq AT$, an expression over A in ordered information systems is defined by $\bigwedge_{a \in A} e(a)$, where $e(a)$ is an atomic expression over a. The set of all expression over A in an OIS is denoted by $E(A)$. For instance, $AT = \{a_1, a_2, a_3\}$, the set of $E(AT)$ is as follows

$$E(\{a_1, a_2, a_3\}) = \{(a_1, \geqslant) \wedge (a_2, \geqslant) \wedge (a_3, \geqslant), (a_1, \geqslant) \wedge (a_2, \geqslant) \wedge (a_3, \leqslant), \ldots, (a_1, \leqslant) \wedge (a_2, \leqslant) \wedge (a_3, \leqslant)\}.$$

In an OIS, $a \in AT$, $v_1 \in V_a$, an atomic formula over a single attribute a ia defined as either $(a, \geqslant, v_1)$ (according to increasing preference) or $(a, \leqslant, v_1)$ (according to decreasing preference). For any $A \subseteq AT$, a formula over A in OIS is defined by $\bigwedge_{a \in A} m(a)$, where $m(a)$ is an atomic formula over a. The set of all formulas over A in an OIS is denoted by $M(A)$. Let the formula $\phi \in M(A)$, $\|\phi\|$ denotes the set of objects satisfying formula ϕ. For example, $(a, \geqslant, v_1)$ and $(a, \leqslant, v_1)$ are atomic formulas, then

$$\|(a, \geqslant, v_1)\| = \{x \in U \mid f(x, a) \geqslant v_1\}$$

and

$$\|(a, \leqslant, v_1)\| = \{x \in U \mid f(x, a) \leqslant v_1\}.$$

However, in an interval ordered information system, the domain of each attribute may be not single-valued but interval. Hence, we modify the definition of a formula over a according to the dominance relation $R_A^{\geqslant}$ as follows

$$\|(a, \trianglerighteq, v_1)\| = \{x \in U \mid f(x, a) \trianglerighteq v_1\}$$

and

$$\|(a, \trianglelefteq, v_1)\| = \{x \in U \mid f(x, a) \trianglelefteq v_1\},$$

where $f(x, a) \trianglerighteq v_1$ denotes that $a^L(x) \geqslant v_1^L$ and $a^U(x) \geqslant v_1^U$, and $f(x, a) \trianglelefteq v_1$ represents that $a^L(x) \leqslant v_1^L$ and $a^U(x) \leqslant v_1^U$, $v_1 = [v_1^L,\ v_1^U]$.

Now we consider an IODT $S = (U, C \cup \{d\}, V, f)$ and a subset of attributes $A \subseteq C$. For two formulas $\phi \in M(A)$ and $\varphi \in M(d)$, a decision rule, denoted by $\phi \to \varphi$, is read "if ϕ then φ." The formula ϕ is called the rule's *antecedent*, and the formula φ is called the rule's *consequent*. We say that an object supports a decision rule if it matches both the condition and the decision parts of the rule. On the other hand, an object is covered by a decision rule if it matches the condition parts of the rule. A decision rule states how "evaluation of objects on attributes A is at least as good as a given level" or "evaluation of objects on attributes A is at most as good as a given level" determines "objects belong (or possibly belong) to at least a given class" or "objects belong (or possibly belong) to at most a given class."

Like dominance rules in [25], there are four types of dominance rules to be considered as follows

(1) certain $\trianglerighteq$-dominance rules with the following syntax:
if $(f(x, a_1) \trianglerighteq v_{a_1}) \wedge (f(x, a_2) \trianglerighteq v_{a_2}) \wedge \cdots \wedge (f(x, a_k) \trianglerighteq v_{a_k}) \wedge (f(x, a_{k+1}) \trianglelefteq v_{a_{k+1}}) \wedge \cdots \wedge (f(x, a_p) \trianglelefteq v_{a_p})$, then $x \in D_i^{\geqslant}$;
(2) possible $\trianglerighteq$-dominance rules with the following syntax:
if $(f(x, a_1) \trianglerighteq v_{a_1}) \wedge (f(x, a_2) \trianglerighteq v_{a_2}) \wedge \cdots \wedge (f(x, a_k) \trianglerighteq v_{a_k}) \wedge (f(x, a_{k+1}) \trianglelefteq v_{a_{k+1}}) \wedge \cdots \wedge (f(x, a_p) \trianglelefteq v_{a_p})$, then x could belong to $D_i^{\geqslant}$;
(3) certain $\trianglelefteq$-dominance rules with the following syntax:
if $(f(x, a_1) \trianglelefteq v_{a_1}) \wedge (f(x, a_2) \trianglelefteq v_{a_2}) \wedge \cdots \wedge (f(x, a_k) \trianglelefteq v_{a_k}) \wedge (f(x, a_{k+1}) \trianglerighteq v_{a_{k+1}}) \wedge \cdots \wedge (f(x, a_p) \trianglerighteq v_{a_p})$, then $x \in D_i^{\leqslant}$; and
(4) possible $\trianglelefteq$-dominance rules with the following syntax:
if $(f(x, a_1) \trianglelefteq v_{a_1}) \wedge (f(x, a_2) \trianglelefteq v_{a_2}) \wedge \cdots \wedge (f(x, a_k) \trianglelefteq v_{a_k}) \wedge (f(x, a_{k+1}) \trianglerighteq v_{a_{k+1}}) \wedge \cdots \wedge (f(x, a_p) \trianglerighteq v_{a_p})$, then x could belong to $D_i^{\leqslant}$;

where $A_1 = \{a_1, a_2, \ldots, a_k\} \subseteq C$, $A_2 = \{a_{k+1}, a_{k+2}, \ldots, a_p\} \subseteq C$, $C = A_1 \cup A_2$, A_1 with increasing preference and A_2 with decreasing preference, and $(v_{a_1}, v_{a_2}, \ldots, v_{a_p}) \in V_{a_1} \times V_{a_2} \times \cdots \times V_{a_p}$, $i \leq r$.

Therefore, in an IODT, for a given upward or downward union $D_i^{\geqslant}$ or $D_j^{\leqslant}$, $i, j \leqslant r$, the rules induced under a hypothesis that objects belonging to $\underline{R_A^{\geqslant}}(D_i^{\geqslant})$ or to $\underline{R_A^{\leqslant}}(D_i^{\leqslant})$ are positive and all the others negative suggest the assignment of an object to "at least class D_i" or to "at most class D_j", respectively. Similarly, the rules induced under a hypothesis that objects belonging to $\overline{R_A^{\geqslant}}(D_i^{\geqslant})$ or to $\overline{R_A^{\leqslant}}(D_i^{\leqslant})$ are positive and all the others negative suggest the assignment of an object could belongs to "at least class D_i" or to "at most class D_j", respectively.

Now we employ an example to illustrate interval ordered decision tables and dominance rules extracted from this type of ODT in the following.

Example 6.1. Let us consider an IODT, constructed from an IOIS in Table 1 and extended by decision attributes d as shown in Table 3.

From Table 2, it is easy to see that $\mathbf{D} = \{D_1, D_2\}$, where

$$D_1 = \{x_5, x_7, x_8, x_9, x_{10}\}, \qquad D_2 = \{x_1, x_2, x_3, x_4, x_6\}.$$

In this interval ordered decision table, because only two decision classes are considered, one can know that $D_1^{\geqslant} = D_1$ and $D_2^{\leqslant} = D_2$. From Example 2.2 and Definition 6.1, we have that

$$\underline{R_C^{\geqslant}}(D_1^{\geqslant}) = \{x_5, x_7, x_8, x_9, x_{10}\},$$

$$\overline{R_C^{\geqslant}}(D_1^{\geqslant}) = \{x_5, x_7, x_8, x_9, x_{10}\} \quad \text{and}$$

$$Bn_C(D_1^{\geqslant}) = \varnothing.$$

Table 3
An interval ordered decision table

U	a_1	a_2	a_3	a_4	a_5	d
x_1	1	[0, 1]	2	1	[1, 2]	1
x_2	[0, 1]	0	[1, 2]	0	1	1
x_3	[0, 1]	0	[1, 2]	1	1	1
x_4	0	0	1	0	1	1
x_5	2	[1, 2]	3	[1, 2]	[2, 3]	2
x_6	[0, 2]	[1, 2]	[1, 3]	[1, 2]	[2, 3]	1
x_7	1	1	2	1	2	2
x_8	[1, 2]	[1, 2]	[2, 3]	2	[2, 3]	2
x_9	[1, 2]	2	[2, 3]	[0, 2]	3	2
x_{10}	2	2	3	[0, 1]	3	2

And, it easily follows from Example 2.2 and Definition 6.2 that

$$\underline{R_C^{\geqslant}}(D_2^{\leqslant}) = \{x_1, x_2, x_3, x_4, x_6\},$$

$$\overline{R_C^{\geqslant}}(D_2^{\leqslant}) = \{x_1, x_2, x_3, x_4, x_6\} \quad \text{and}$$

$$Bn_C(D_2^{\leqslant}) = \varnothing.$$

Therefore, we can obtain the following set of dominance rules from the considered interval ordered decision table:

r_1 : $(a_1, \trianglerighteq, 1) \wedge (a_2, \trianglerighteq, 1) \wedge (a_3, \trianglerighteq, 2) \wedge (a_4, \trianglerighteq, 1) \wedge (a_5, \trianglerighteq, 2) \rightarrow (d, \geqslant, 2)$ //supported by objects x_5, x_7, x_8;

r_2 : $(a_1, \trianglerighteq, [1, 2]) \wedge (a_2, \trianglerighteq, 2) \wedge (a_3, \trianglerighteq, [2, 3]) \wedge (a_4, \trianglerighteq, [0, 2]) \wedge (a_5, \trianglerighteq, 3) \rightarrow (d, \geqslant, 2)$ //supported by objects x_9;

r_3 : $(a_1, \trianglerighteq, 2) \wedge (a_2, \trianglerighteq, 2) \wedge (a_3, \trianglerighteq, 3) \wedge (a_4, \trianglerighteq, [0, 1]) \wedge (a_5, \trianglerighteq, 3) \rightarrow (d, \geqslant, 2)$ //supported by objects x_{10};

r_4 : $(a_1, \trianglelefteq, 1) \wedge (a_2, \trianglelefteq, [0, 1]) \wedge (a_3, \trianglelefteq, 2) \wedge (a_4, \trianglelefteq, 1) \wedge (a_5, \trianglelefteq, [1, 2]) \rightarrow (d, \leqslant, 1)$ //supported by objects x_1, x_2, x_3, x_4;

and

r_5 : $(a_1, \trianglelefteq, [0, 2]) \wedge (a_2, \trianglelefteq, [1, 2]) \wedge (a_3, \trianglelefteq, [1, 3]) \wedge (a_4, \trianglelefteq, [1, 2]) \wedge (a_5, \trianglelefteq, [2, 3]) \rightarrow (d, \leqslant, 1)$ //supported by objects x_2, x_3, x_4, x_6

where the rules r_1, r_2, r_3 are certain $\trianglerighteq$-dominance rules and the rules r_4, r_5 are certain $\trianglelefteq$-dominance rules. □

For any dominance rule $r : \phi \rightarrow \varphi$, its certainty factor, support factor and coverage factor can be defined respectively as follows:

$$\text{cer}(\phi \rightarrow \varphi) = \frac{\text{card}(\|\phi \wedge \varphi\|)}{\text{card}(\|\phi\|)},$$

$$\text{sup}(\phi \rightarrow \varphi) = \frac{\text{card}(\|\phi \wedge \varphi\|)}{\text{card}(|U|)}$$

and

$$\text{cov}(\phi \rightarrow \varphi) = \frac{\text{card}(\|\phi \wedge \varphi\|)}{\text{card}(\|\varphi\|)}.$$

The certainty factor can be interpreted as the frequency of objects having the property φ in the set of objects having the property ϕ and the coverage factor as the frequency of objects having the property ϕ in the set of objects having the property φ. And, the support factor denotes the probability of objects having both the property ϕ and the property φ within the universe U.

Example 6.2. Compute three factors of the dominance rule r_1 in Example 6.1.

$$r_1 : (a_1, \trianglerighteq, 1) \wedge (a_2, \trianglerighteq, 1) \wedge (a_3, \trianglerighteq, 2) \wedge (a_4, \trianglerighteq, 1) \wedge (a_5, \trianglerighteq, 2) \rightarrow (d, \trianglerighteq, 2).$$

Computing these factors, we have that $\text{card}(\|\phi\|) = 3$, $\text{card}(\|\phi \wedge \varphi\|) = 3$, $\text{card}(\|\varphi\|) = 5$ and $|U| = 10$; thus $\text{cer}(r_1) = 1$, $\text{sup}(r_1) = 0.3$ and $\text{cov}(r_1) = 0.6$.

7. Attribute reduction in IODT

To extract more briefer dominance rules it is necessary to reduce some dispensable attributes in the condition part of a given interval ordered decision table. In this section, an practical approach to attribute reduction in interval ordered decision tables is established and an illustrative example is employed to show its mechanism.

Let $S = (U, C \cup \{d\}, V, f)$ be an IODT and d is an overall preference of objects. Denoted by

$$R_{\{d\}}^{\geqslant} = \{(x, y) : f(x, d) \geqslant f(y, d)\},$$

where $R_{\{d\}}^{\geqslant}$ is a dominance relation of decision attribute d. If $R_C^{\geqslant} \subseteq R_{\{d\}}^{\geqslant}$, then S is called *consistent*; otherwise it is called *inconsistent*. For example, Table 3 is a consistent interval ordered decision table, where $C = \{a_1, a_2, a_3, a_4, a_5\}$ is the

Table 4
The discernibility matrix of Table 3

x_i/x_j	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$a_1\cdots a_5$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
x_2	$a_1\cdots a_5$	a_4	$\varnothing$	$a_1\cdots a_5$	$\varnothing$	$a_1\cdots a_5$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
x_3	$a_1a_2a_3a_5$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$a_1\cdots a_5$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
x_4	$a_1\cdots a_5$	a_1a_3	$a_1a_3a_4$	$\varnothing$	$\varnothing$	$a_1\cdots a_5$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
x_5	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	a_4	a_2a_5	a_2a_5
x_6	a_1a_3	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
x_7	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$a_1\cdots a_5$	$a_1\cdots a_5$	$\varnothing$	$a_1\cdots a_5$	$a_1\cdots a_5$	$a_1a_2a_3a_5$
x_8	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	a_1a_3	$\varnothing$	$\varnothing$	$\varnothing$	a_2a_5	$a_1a_2a_3a_5$
x_9	a_4	$\varnothing$	a_4	$\varnothing$	$a_1a_3a_4$	a_4	a_4	a_4	$\varnothing$	a_1a_3
x_{10}	a_4	$\varnothing$	a_4	$\varnothing$	a_4	a_4	a_4	a_4	a_4	$\varnothing$

condition attribute set and $\{d\}$ is the decision attribute set. In other words, any pair $(x, y) \in R_C^{\geqslant}$ belongs to the relation $R_{\{d\}}^{\geqslant}$ in this ordered decision table. In this paper, we only deal with attribute reduction of a consistent IODT.

Definition 7.1. Let $S = (U, C \cup \{d\}, V, f)$ be a consistent IODT and $A \subseteq C$. If $R_A^{\geqslant} \subseteq R_{\{d\}}^{\geqslant}$ and $R_B^{\geqslant} \not\subseteq R_{\{d\}}^{\geqslant}$ for any $B \subset A$, then we call A a *relative attribute* reduction of S.

Similar to the idea of reducts of incomplete ODT in [25], we denote by $D^* = \{(x, y) : f(x, d) < f(y, d)\}$, and denote by

$$\mathrm{Dis}^*(x, y) = \begin{cases} \{a \in C : (x, y) \notin R_{\{a\}}^{\geqslant}\}, & (x, y) \notin D^*; \\ \varnothing, & (x, y) \in D^*. \end{cases}$$

Then $\mathrm{Dis}^*(x, y)$ is called a *discernibility set* for objects x and y and $\mathbf{Dis}^* = (\mathrm{Dis}^*(x, y) : x, y \in U)$ is called a *discernibility matrix* for this IODT.

Similar to complete ordered decision tables and incomplete ordered decision tables, we can get the following property.

Property 7.1. *Let $S = (U, C \cup \{d\}, V, f)$ be an IODT, $A \subseteq C$ and $\mathrm{Dis}^*(x, y)$ the discernibility attributes set of S with respect to $R_{\{d\}}^{\geqslant}$. Then, $R_A^{\geqslant} \subseteq R_{\{d\}}^{\geqslant}$ iff $A \cap \mathrm{Dis}^*(x, y) \neq \varnothing$ ($\mathrm{Dis}^*(x, y) \neq \varnothing$).*

Proof. This proof is similar to the proof of Property 5.2. □

Definition 7.2. Let $S = (U, C \cup \{d\}, V, f)$ be an IODT, $A \subseteq C$ and $\mathrm{Dis}^*(x, y)$ the discernibility attributes set of S with respect to $R_{\{d\}}^{\geqslant}$. Denoted by

$$M^* = \bigwedge \left\{ \bigvee \{a : a \in \mathrm{Dis}^*(x, y)\} : x, y \in U \right\},$$

then M^* is referred to as a discernibility function.

From the definition of discernibility function, we can design a practical approach to relative attribute reduction in an IODT as follows.

Property 7.2. *Let $S = (U, C \cup \{d\}, V, f)$ be an IODT. The minimal disjunctive normal form of discernibility function M^* is*

$$M^* = \bigvee_{k=1}^{t} \left(\bigwedge_{s=1}^{q_k} a_{i_s} \right).$$

Denoted by $B_k = \{a_{i_s} : s = 1, 2, \ldots, q_k\}$, then $\{B_k : k = 1, 2, \ldots, t\}$ are the set of all relative attribute reductions of this system.

Proof. It follows directly from Property 7.1 and the definition of minimal disjunctive normal form of the discernibility function. □

Property 7.2 provides a practical approach to relative attribute reduction in an interval ordered decision table.

Example 7.1. Compute all relative attribute reductions in Table 3.

Table 4 is a discernibility matrix of this consistent ordered decision table, where values of $\mathrm{Dis}^*(x_i, x_j)$ for any pair (x_i, x_j) of objects from U are placed.

From Table 4, one can obtain that

$$\begin{aligned} M^* &= (a_1 \vee a_2 \vee a_3 \vee a_4 \vee a_5) \wedge a_4 \wedge (a_2 \vee a_5) \wedge (a_1 \vee a_2 \vee a_3 \vee a_5) \wedge (a_1 \vee a_3) \wedge (a_1 \vee a_3 \vee a_4) \\ &= (a_2 \vee a_5) \wedge (a_1 \vee a_3) \wedge a_4 \\ &= (a_1 \wedge a_2 \wedge a_4) \vee (a_1 \wedge a_4 \wedge a_5) \vee (a_2 \wedge a_3 \wedge a_4) \vee (a_3 \wedge a_4 \wedge a_5). \end{aligned}$$

Table 5
An interval ordered decision table about venture investment [29]

Projects	Market	Technology	Management	Environment	Production	Finance	Venture
x_1	[3, 4]	[4, 5]	[3, 4]	[3, 4]	[2, 3]	[4, 5]	High
x_2	[1, 2]	[1, 2]	[1, 3]	[1, 3]	[2, 3]	[1, 3]	Low
x_3	[3, 4]	[4, 5]	[3, 5]	[3, 4]	[3, 5]	[4, 5]	High
x_4	[2, 3]	[4, 5]	[2, 3]	[2, 4]	[2, 3]	[3, 5]	High
x_5	[1, 2]	[1, 2]	[1, 3]	[1, 2]	[2, 3]	[1, 3]	Low

Hence, there are four relative attribute reductions $\{a_1, a_2, a_4\}$, $\{a_1, a_4, a_5\}$, $\{a_2, a_3, a_4\}$ and $\{a_3, a_4, a_5\}$ in this consistent interval ordered decision table. From this result, we know that the condition attribute a_4 is indispensable for this decision table. Through a relative attribute reduction, one can obtain more briefer dominance rules. For example, by taking the relative attribute reduction $\{a_1, a_2, a_4\}$, the five dominance rules in Example 6.1 can be simply represented as follows.

$r_1 : (a_1, \trianglerighteq, 1) \wedge (a_2, \trianglerighteq, 1) \wedge (a_4, \trianglerighteq, 1) \rightarrow (d, \geqslant, 2)$ //supported by objects x_5, x_7, x_8;
$r_2 : (a_1, \trianglerighteq, [1, 2]) \wedge (a_2, \trianglerighteq, 2) \wedge (a_4, \trianglerighteq, [0, 2]) \rightarrow (d, \geqslant, 2)$ //supported by objects x_9;
$r_3 : (a_1, \trianglerighteq, 2) \wedge (a_2, \trianglerighteq, 2) \wedge (a_4, \trianglerighteq, [0, 1]) \rightarrow (d, \geqslant, 2)$ //supported by objects x_{10};
$r_4 : (a_1, \trianglelefteq, 1) \wedge (a_2, \trianglelefteq, [0, 1]) \wedge (a_4, \trianglelefteq, 1) \rightarrow (d, \leqslant, 1)$ //supported by objects x_1, x_2, x_3, x_4; and
$r_5 : (a_1, \trianglelefteq, [0, 2]) \wedge (a_2, \trianglelefteq, [1, 2]) \wedge (a_4, \trianglelefteq, [1, 2]) \rightarrow (d, \leqslant, 1)$ //supported by objects x_2, x_3, x_4, x_6
where the rules r_1, r_2, r_3 are certain $\trianglerighteq$-dominance rules and the rules r_4, r_5 are certain $\trianglelefteq$-dominance rules. □

8. Case study

Venture capital has become an increasingly important source of financing for new companies, particularly when such companies are operating on the frontier of emerging technologies and markets. It plays an essential role in the entrepreneurial process [36]. For an investor or decision maker, he may need to adopt a better one from some possible investment projects or find some directions from existing successful investment projects before investing. The purpose of this section is, through a venture investment issue, to illustrate how to make a decision by using the approaches proposed in this paper.

Let us consider an investment issue of a venture investment company [29]. There are five investment projects x_i $(i = 1, 2, \ldots, 5)$ can be considered. They can be evaluated from the view of venture factors. Venture factors are classified into six factors, which are market venture, technology venture, management venture, environment venture, production venture and finance venture. These six factors are all increasing preference and the value of each project under each factor is given by a evaluation expert through an interval number. Table 5 is a evaluation table about venture investment given by a expert [29], where $U = \{x_1, x_2, x_3, x_4, x_5\}$, $C = \{$Market, Technology, Management, Environment, Production, Finance$\}$ and $d = \{$Venture$\}$. For convenience, in the sequel, M_1, T, M_2, E, P, F and V will stand for Market, Technology, Management, Environment, Production, Finance and Venture, respectively.

From Table 5, we have that

$$U/R_C^{\geqslant} = \{[x_1]_C^{\geqslant},\ [x_2]_C^{\geqslant},\ [x_3]_C^{\geqslant},\ [x_4]_C^{\geqslant},\ [x_5]_C^{\geqslant}\},$$

where $[x_1]_C^{\geqslant} = \{x_1, x_3\}$, $[x_2]_C^{\geqslant} = \{x_1, x_2, x_3, x_4\}$, $[x_3]_{AT}^{\geqslant} = \{x_3\}$, $[x_4]_{AT}^{\geqslant} = \{x_1, x_3, x_4\}$ and $[x_5]_{AT}^{\geqslant} = \{x_1, x_2, x_3, x_4, x_5\}$.

From the definition of dominance degree, we can get the dominance relation matrix of this table with respect to $U/R_C^{\geqslant}$ as

$$\begin{pmatrix} 1 & 1 & 0.8 & 1 & 1 \\ 0.6 & 1 & 0.4 & 0.8 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0.8 & 1 & 0.6 & 1 & 1 \\ 0.4 & 0.8 & 0.2 & 0.6 & 1 \end{pmatrix}.$$

Therefore, one can obtain that

$$D_C(x_1) = 0.95, \quad D_C(x_2) = 0.70, \quad D_C(x_3) = 1.00, \quad D_C(x_4) = 0.85 \quad \text{and} \quad D_C(x_5) = 0.60.$$

In what follows, we rank these five projects according to the number of $D_C(x_i)$. A project with whole dominance degree implies that it has higher investment venture.

$$x_3 \succcurlyeq x_1 \succcurlyeq x_4 \succcurlyeq x_2 \succcurlyeq x_5.$$

Thus, the investment venture of project x_3 is highest and that of project x_5 is lowest. The decision maker may select the project x_5 to invest.

From Table 5, it is easy to see that $\mathbf{d} = \{D_1, D_2\}$, where

$$D_1 = \{x_2, x_5\}, \qquad D_2 = \{x_1, x_3, x_4\}.$$

Table 6
The discernibility matrix of Table 5

x_i/x_j	x_1	x_2	x_3	x_4	x_5
x_1	$\varnothing$	$\varnothing$	$\{M_2, P\}$	$\varnothing$	$\varnothing$
x_2	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
x_3	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
x_4	$\{M_1, M_2, E, F\}$	$\varnothing$	$\{M_1, M_2, E, F\}$	$\varnothing$	$\varnothing$
x_5	$\varnothing$	$\{E\}$	$\varnothing$	$\varnothing$	$\varnothing$

In this interval ordered decision table, because only two decision classes are considered, one can know that $D_1^{\geqslant} = D_1$ and $D_2^{\leqslant} = D_2$. From Definition 6.1, we have that

$$\underline{R_C^{\geqslant}}(D_1^{\geqslant}) = \{x_1, x_3, x_4\}, \qquad \overline{R_C^{\geqslant}}(D_1^{\geqslant}) = \{x_1, x_3, x_4\} \quad \text{and} \quad Bn_C(D_1^{\geqslant}) = \varnothing.$$

And, we have that

$$U/R_C^{\leqslant} = \{[x_1]_C^{\leqslant},\ [x_2]_C^{\leqslant},\ [x_3]_C^{\leqslant},\ [x_4]_C^{\leqslant},\ [x_5]_C^{\leqslant}\},$$

where $[x_1]_C^{\leqslant} = \{x_1, x_2, x_4, x_5\}$, $[x_2]_C^{\leqslant} = \{x_2, x_5\}$, $[x_3]_{AT}^{\leqslant} = \{x_1, x_2, x_3, x_4, x_5\}$, $[x_4]_{AT}^{\leqslant} = \{x_2, x_4, x_5\}$ and $[x_5]_{AT}^{\leqslant} = \{x_5\}$.

Thus, it easily follows from Definition 6.2 that

$$\underline{R_C^{\geqslant}}(D_2^{\leqslant}) = \{x_2, x_5\}, \qquad \overline{R_C^{\geqslant}}(D_2^{\leqslant}) = \{x_2, x_5\} \quad \text{and} \quad Bn_C(D_2^{\leqslant}) = \varnothing.$$

Therefore, we can obtain the following set of dominance rules from the considered interval ordered decision table:

r_1 : $(M_1, \trianglerighteq, [2, 3]) \wedge (T, \trianglerighteq, [4, 5]) \wedge (M_2, \trianglerighteq, [2, 3]) \wedge (E, \trianglerighteq, [2, 4]) \wedge (P, \trianglerighteq, [2, 3]) \wedge (F, \trianglerighteq, [3, 5]) \rightarrow (V, \geqslant, \text{High})$ //supported by objects x_1, x_3, x_4;

r_2 : $(M_1, \trianglelefteq, [1, 2]) \wedge (T, \trianglelefteq, [1, 2]) \wedge (M_2, \trianglelefteq, [1, 3]) \wedge (E, \trianglelefteq, [1, 3]) \wedge (P, \trianglelefteq, [2, 3]) \wedge (F, \trianglelefteq, [1, 3]) \rightarrow (V, \leqslant, \text{Low})$ //supported by objects x_2, x_5.

To extract much simpler dominance rules, we compute relative attribute reductions of this decision table. From the definition of consistency of a decision table, one can know that Table 5 is consistent. Therefore, the relative attribute reductions of this decision table can be obtained by the proposed attribute-reduction approach in Section 7. Table 6 is the discernibility matrix of this consistent decision table, where values of $\text{Dis}^*(x_i, x_j)$ for any pair (x_i, x_j) of projects are placed.

From Table 6, one can obtain that

$$\begin{aligned} M^* &= (M_2 \vee P) \wedge (M_1 \vee M_2 \vee E \vee F) \wedge E \\ &= (M_2 \wedge E) \vee (P \wedge E). \end{aligned}$$

Hence, there are two relative attribute reductions in this consistent interval ordered decision table about venture investment, which are {Management, Environment} and {Production, Environment}. From this result, we know that the venture factor Environment is indispensable for this decision problem. Through these two relative attribute reductions, one can obtain two sets of more briefer dominance rules. By taking the relative attribute reduction {Management, Environment}, the two dominance rules in the above part can be simply represented as follows:

r_1' : $(M_2, \trianglerighteq, [2, 3]) \wedge (E, \trianglerighteq, [2, 4]) \rightarrow (V, \geqslant, \text{High})$//supported by objects x_1, x_3, x_4;

r_2' : $(M_2, \trianglelefteq, [1, 3]) \wedge (E, \trianglelefteq, [1, 3]) \rightarrow (V, \leqslant, \text{Low})$//supported by objects x_2, x_5.

where the rule r_1' is a certain $\trianglerighteq$-dominance rule and the rule r_2' is a certain $\trianglelefteq$-dominance rule. And, by taking the relative attribute reduction {Production, Environment}, the two dominance rules in the above part can be simply represented as follows:

r_1'' : $(E, \trianglerighteq, [2, 4]) \wedge (P, \trianglerighteq, [2, 3]) \rightarrow (V, \geqslant, \text{High})$ //supported by objects x_1, x_3, x_4;

r_2'' : $(E, \trianglelefteq, [1, 3]) \wedge (P, \trianglelefteq, [2, 3]) \rightarrow (V, \leqslant, \text{Low})$ //supported by objects x_2, x_5

where the rule r_1'' is a certain $\trianglerighteq$-dominance rule and the rule r_2'' is a certain $\trianglelefteq$-dominance rule. Therefore, management venture, environment venture and production venture are three important venture factors for this investment issue.

9. Conclusions

Rough set theory has been proved to be a useful mathematical tool for classification and prediction. However, as many real-world problems deal with ordering objects instead of classifying objects, one of the extensions of the classical rough set approach is the dominance-based rough set approach, which is mainly based on substitution of the indiscernibility relation by a dominance relation. Interval information systems are an important type of data tables, which are generalized models of single-valued information systems. We deal with interval ordered information systems and interval ordered decision tables in present research.

In this paper, we have introduced a dominance relation to interval information systems and have given a ranking method for all objects by using whole dominance degree of each object. Based on this dominance relation, we have established a rough set approach in this type of OIS, which is mainly based on substitution of the indiscernibility relation by the dominance relation $R_A^{\geqslant}$. For extracting dominance rules, we have discussed interval ordered decision tables and dominance rules extracted from this types of decision tables. In order to extract much simpler dominance rules, based on the discernibility matrices, we have proposed attribute reductions of interval ordered information systems and interval ordered decision tables that eliminate only information that are not essential from the view of the ordering of objects or dominance rules. The approaches show how to simplify an interval ordered information system and find much simpler dominance rules directly from an interval ordered decision table.

Acknowledgements

This work was partially supported by the high technology research and development program (No. 2007AA01Z165), the national natural science foundation of China (No. 60773133, No. 70471003, No. 60573074), the foundation of doctoral program research of the ministry of education of China (No. 20050108004) and key project of science and technology research of the ministry of education of China.

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