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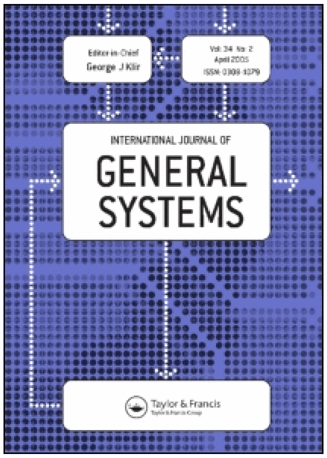

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Study of decision implications based on formal concept analysis

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In this paper, the notions of decision table and decision rule in Rough Set Theory are introduced naturally into Formal Concept Analysis as decision context and decision implication. Since extracting decision implications directly from decision context takes time, we present an inference rule called α-decision inference rule to eliminate the superfluous decision implications. Moreover, based on the inference rule we introduce the notion of α-maximal decision implication and prove that the set of all α-maximal decision implications is complete (α-complete) and non-redundant (α-non-redundant). Finally, we present a method to generate the set.

Keywords: *Formal concept analysis; Concept lattice; Rough set theory; Decision context; Decision implication*

1. Introduction

Formal concept analysis (FCA) is an order-theoretic method for the mathematical analysis of scientific data, pioneered by Wille (1982) in mid 80s. Over the past twenty years, FCA has been widely studied (Ganter and Wille 1999, Qu *et al.* 2004) and become a powerful tool for machine learning (Zupa and Bohance 1999), software engineering (Tonella 2003, Dekel 2003) and information retrieval (Carpineto and Romano 2004).

In essence, FCA is based on a formalization of the philosophical understanding of a *concept* as a unit of thought constituted by its *extent* and *intent*. The extent of a concept is understood as the collection of all objects belonging to the concept and the intent as the multitude of all attributes common to all those objects. The transformation from two-dimensional incidence tables to *concept lattices* structure is a crucial keystone from which

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FCA derives much of its power and versatility as a modelling tool. The concept lattices obtained the way turn out to be exactly the complete lattices, and the particular way in which they structure and represent knowledge is very appealing and natural from the perspective of many scientific disciplines.

On the other hand, Rough Set Theory (Pawlak 1991) has attracted attention of many researchers and practitioners, who contributed essentially to its development and applications. In Rough Set, each row of a decision table determines a decision rule, which specifies decisions that should be taken when conditions pointed out by condition attributes are satisfied (Pawlak 1991).

However, there has been only little work relating decision rule in FCA. The aim of this paper is to introduce decision context and decision implication (rather than decision rule in Rough Set Theory) to FCA. The organization of the paper is as follows. In Section 2, we recall the fundamental notions and results. Section 3 is devoted to introducing the notions of decision context and decision implication. Section 4 serves to present the system of α-maximal decision implications. The method to generate decision implications is proposed in Section 5. Section 6 contains an illustrative example. Conclusion and discussion of further work will close the paper in Section 7.

2. Basic notions of FCA

This section provides a brief overview over FCA, in order to allow for a better understanding for the overall picture. We introduce the most basic notions of FCA, namely formal contexts, formal concepts, concept lattices and implications. For more extensive introduction refer to Ganter and Wille (1999).

In FCA, an elementary form of the representation of data is defined mathematically as *formal context*.

DEFINITION 1. A formal context is a triple $K = (G, M, I)$, where G and M are sets, and $I \subseteq G \times M$ is a binary relation. In the case, the members of G are called objects, the members of M are called attributes, and I is viewed as an incidence relation between objects and attributes. Accordingly, we write gIm or $(g, m) \in I$ expressing "the object g has the attribute m".

Formal contexts are mostly represented by rectangular tables and an example is illustrated by Table 1, the rows of which are headed by the object names and the columns headed by the attribute names. In the table, a cross means that the row object has the column attribute.

DEFINITION 2. For a set $A \subseteq G$ of objects we define:

$$A^I = \{m \in M | gIm, \ \ \forall g \in A\}$$

(the set of attributes common to the objects in A). Correspondingly, for a set $B \subseteq M$ we define:

$$B^I = \{g \in G | gIm, \ \ \forall m \in B\}$$

(the set of objects which have all attributes in B).

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Table 1. A context K.

	a_1	a_2	a_3	a_4	a_5	a_6	d_1	d_2
x_1	×	×			×	×		
x_2			×					×
x_3	×		×			×	×	
x_4		×						
x_5		×	×	×	×	×	×	×
x_6			×		×			
x_7		×	×	×		×		×
x_8	×	×		×	×	×	×	

Definition 3. Let $K = (G, M, I)$ be a formal context, $A \subseteq G,\ \ B \subseteq M$. A pair $\mathfrak{C} = (A, B)$ is called a formal concept of K, if $A^I = B,\ \ B^I = A$. In the case, A is the intent of $\mathfrak{C}$ and B is the extent of $\mathfrak{C}$. $\mathfrak{B}(K)$ denotes the set of all concepts of the context K.

The description of a concept by extent and intent is redundant, because each of the two parts determines the other. But for many reasons this redundant description is very convenient.

Formal concepts can be (partially) ordered in a natural way. Again, the definition is inspired by the way we usually order concepts in a subconcept-superconcept hierarchy: "pig" is a subconcept of "mammal", because every pig is a mammal. Transferring this to formal concepts, the natural definition is as follows.

Definition 4. Let $K = (G, M, I)$ be a formal context, $\mathfrak{C}_1 = (A_1, B_1),\ \ \mathfrak{C}_2 = (A_2, B_2) \in \mathfrak{B}(K)$. We define:

$$\mathfrak{C}_1 \preceq \mathfrak{C}_2 \Leftrightarrow A_1 \subseteq A_2 (\Leftrightarrow B_1 \supseteq B_2).$$

In the case, $\mathfrak{C}_2$ is a superconcept of $\mathfrak{C}_1$ and $\mathfrak{C}_1$ is a subconcept of $\mathfrak{C}_2$. The relation " $\preceq$ " is called the hierarchical order of the concepts. The set of all concepts ordered in the way is called the concept lattice of the context K.

One of the aspects of FCA thus is attribute logic, the study of possible attribute combinations. Dependencies between the attributes can be described by implications. An implication between attributes in M is a pair of subsets of M, denoted by $B_1 \rightarrow B_2$. The set B_1 is the premise of the implication $B_1 \rightarrow B_2$, and B_2 is its conclusion. Formally,

Definition 5. Let $K = (G, M, I)$ be a formal context, $B_1, B_2 \subseteq M$. $B_1 \rightarrow B_2$ is true if each object which has all attributes from B_1 has also all attributes from B_2. In the case, $B_1 \rightarrow B_2$ also is called an implication in the context K.

We have the following simple facts (Ganter and Wille 1999).

Theorem 1. If $K = (G, M, I)$ is a formal context, A, A_1, A_2 are sets of objects and B, B_1, B_2 are set of attributes, then:

$$\begin{array}{ll}
1)\ A_1 \subseteq A_2 \Rightarrow A_2^I \subseteq A_1^I & 1')\ B_1 \subseteq B_2 \Rightarrow B_2^I \subseteq B_1^I \\
2)\ A \subseteq A^{II} & 2')\ B \subseteq B^{II} \\
3)\ A^I = A^{III} & 3')\ B^I = B^{III} \\
4)\ A \subseteq B^I \Leftrightarrow B \subseteq A^I &
\end{array}$$

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3. Decision context and decision implication

In the section, we distinguish in a formal context two classes of attributes, called condition and decision attributes as called in Rough Set Theory.

DEFINITION 6. Let $K = (G, M, I)$ be a formal context. The context is called a decision context if $M = C \cup D$, $C \cap D = \emptyset$ and $I = I_C \cup I_D$, where C is the set of condition attributes, D is the set of decision attributes, $I_C \subseteq G \times C$ is the set of condition incidence relations, and $I_D \subseteq G \times D$ is the set of decision incidence relations.

In fact, a decision context consists of two sub-contexts, namely the condition sub-context $K_C = (G, C, I_C)$ and the decision sub-context $K_D = (G, D, I_D)$. For $A \subseteq G$, $B_{C1} \subseteq C$ and $B_{D1} \subseteq D$, by Definition 2, the symbols $A^{I_C}, A^{I_D}, B_{C1}^{I_C}, B_{D1}^{I_D}$ can be abbreviated to $A^C, A^D, B_{C1}^C, B_{D1}^D$.

DEFINITION 7. Let $K = (G, C \cup D, I_C \cup I_D)$ be a decision context. The decision context is consistent if for all $g, h \in G$, $g^{CC} = h^{CC}$ implies $g^{DD} = h^{DD}$.

The definition 7 expresses that, in a consistent decision context, if two objects posses the same set of condition attributes(i.e. $g^{CC} = h^{CC}$), then their decision attributes are same, that is to say, when we will make some decisions, in the face of same conditions, we will not make different decisions. Throughout our paper, we assume that all contexts we deal with are consistent decision contexts.

DEFINITION 8. Let $K = (G, C \cup D, I_C \cup I_D)$ be a decision context, and B_1, B_2 are sets of attributes. An implication $B_1 \rightarrow B_2$ of the formal context K is called a decision implication of the decision context K, if $B_1 \subseteq C$ and $B_2 \subseteq D$.

THEOREM 2. Let $K = (G, C \cup D, I_C \cup I_D)$ be a decision context, $B_{C1} \subseteq C$ and $B_{D1} \subseteq D$. Then $B_{C1} \rightarrow B_{D1}$ is a decision implication if and only if $B_{C1}^C \subseteq B_{D1}^D$.

Proof. One can easily see that $B_{C1}^C = B_{C1}^I$ (and the same for D). Therefore, $B_{C1}^C \subseteq B_{D1}^D$ iff $B_{C1}^I \subseteq B_{D1}^I$ which is a well-known condition for $B_{C1} \rightarrow B_{D1}$ being true in K. □

Note that, when we refer to a context K or simply K, we mean a formal context K or a decision context K, depending on whether the form $B_1 \rightarrow B_2$ is an implication or a decision implication.

4. The system of α-maximal decision implications

Given a decision context $K = (G, C \cup D, I_C \cup I_D)$, we can find all decision implications as follows: for any subset $B_{C1} \subseteq C$ and any $B_{D1} \subseteq D$, verify the correctness of the formula $B_{C1}^C \subseteq B_{D1}^D$ by Theorem 2; the correctness of $B_{C1}^C \subseteq B_{D1}^D$ implies that, the decision implication $B_{C1} \rightarrow B_{D1}$ holds in the decision context K.

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In general, however, the number of decision implications in a decision context is quite large and in a given set of implications, there are lots of redundant implications, which can be deduced from other implications by means of so-called inference rules.

THEOREM 3 (α-DECISION INFERENCE RULE). Let $K = (G, C \cup D, I_C \cup I_D)$ be a decision context, $B_{C1}, B_{C2} \subseteq C$, $B_{D1}, B_{D2} \subseteq D$ and $B_{C1} \rightarrow B_{D1}$ is a decision implication of K. If $B_{C2} \supseteq B_{C1}$ and $B_{D1} \supseteq B_{D2}$, then $B_{C2} \rightarrow B_{D2}$ is a decision implication of K.

Proof. See (Ganter and Wille 1999). □

The above theorem, from the logistic angle (Hamilton 1978), can be called the Soundness Theorem of α-Decision Inference Rule. The inference rule can be characterized by the following form:

$$\frac{B_{C1} \rightarrow B_{D1}, B_{C2} \supseteq B_{C1}, B_{D1} \supseteq B_{D2}}{B_{C2} \rightarrow B_{D2}}$$

which means that, if $B_{C1} \rightarrow B_{D1}$ and $B_{C2} \supseteq B_{C1}, B_{D1} \supseteq B_{D2}$, then $B_{C2} \rightarrow B_{D2}$. In other words, the decision implication $B_{C2} \rightarrow B_{D2}$ can be inferred from the implication $B_{C1} \rightarrow B_{D1}$ by α-decision inference rule. Certainly, there are lots of such inference rule by which we can eliminate the superfluous decision implications. Compared to other inference rules, however, the α-decision inference rule is, perhaps, most intuitive and effective, as showed by our illustrative example in Section 6.

DEFINITION 9. Let $K = (G, C \cup D, I_C \cup I_D)$ be a decision context, Σ be a set of decision implications and $B_{C1} \rightarrow B_{D1}$ is a decision implication of K. If the decision implication $B_{C1} \rightarrow B_{D1}$ can be inferred from the set Σ by some inference rule τ, we say the implication $B_{C1} \rightarrow B_{D1}$ can be τ-inferred from Σ. In this case, we call the implication $B_{C1} \rightarrow B_{D1}$ τ-redundant for Σ. Furthermore, if all decision implications of K can be τ-inferred from Σ, we say, the decision implication set Σ is τ-complete.

DEFINITION 10. Let $K = (G, C \cup D, I_C \cup I_D)$ be a decision context. An decision implication $B_{C1} \rightarrow B_{D1}$ is α-maximal if

(1) for any $B_{C2} \subset B_{C1}$, $B_{C2} \rightarrow B_{D1}$ does not hold in the decision context K; and
(2) for any $B_{D1} \subset B_{D2}$, $B_{C1} \rightarrow B_{D2}$ does not hold in the decision context K.

In this case, we also say that $B_{C1} \rightarrow B_{D1}$ is a α-maximal decision implication of K.

THEOREM 4. The set Σ of all α-maximal decision implications is α-non-redundant, i.e. for any α-maximal decision implication $B_{C1} \rightarrow B_{D1} \in \Sigma$, it can not be α-inferred from the set $\Sigma \backslash (B_{C1} \rightarrow B_{D1})$.

Proof. The statement follows immediately from Definition 10. □

Theorem 4 shows that the α-maximal decision implications are independent with each other. And the following theorem, also from the logistic angle, can be called the Adequacy

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Theorem for the System of α-Maximal Decision Implications, showing that the system of α-maximal decision implications is relatively enough for the system of all the decision implications.

THEOREM 5. Let $K=(G, C\cup D, I_C\cup I_D)$ be a decision context and Σ denotes the set of all α-maximal decision implication. Then, Σ is α-complete.

Proof. Let $B_{C2}\rightarrow B_{D2}$ be a decision implication of K. Then there exists a minimal set $B_{C1}\subseteq B_{C2}$ such that $B_{C1}\rightarrow B_{D2}$ is true. In the same way, there must exist a maximal set $B_{D1}\supseteq B_{D2}$ such that $B_{C1}\rightarrow B_{D1}$ is true. It is to see that $B_{C1}\rightarrow B_{D1}$ is α-maximal and that $B_{C2}\rightarrow B_{D2}$ can be α-inferred by $B_{C1}\rightarrow B_{D1}$. That means that Σ is α-complete. □

5. Generating the set of all α-maximal decision implications

In the section, we will generate the set of all α-maximal decision implications by means of minimal generators(i.e. key sets) (Stumme *et al.* 2000). First, we present a theorem used later.

THEOREM 6. Let $K=(G, C\cup D, I_C\cup I_D)$ be a decision context, $B_{C1}\subseteq C$. Then the implication $B_{C1}\rightarrow B_{C1}^{CD}$ is a decision implication of K.

Proof. Since $B_{C1}^{C}\subseteq G$, by Theorem 1, we have $B_{C1}^{C}\subseteq B_{C1}^{CDD}$, which, by Theorem 2, completes the proof. □

DEFINITION 11. Let $K=(G, M, I)$ be a formal context, $\mathfrak{C}=(A,B)\in\mathfrak{B}(K)$, and $X\subseteq A$. X is called a minimal generator of the concept $\mathfrak{C}$, if $X^{II}=A$ and $Y^{II}\subset X^{II}$, for all $Y\subset X$.

A minimal generator of a concept is the minimal information that permits to retrieve a concept. The following theorem expresses that, the premise of a α-maximal decision implication must be a minimal generator of one concept in the condition sub-context.

THEOREM 7. Let $K=(G, C\cup D, I_C\cup I_D)$ be a decision context and $B_{C1}\rightarrow B_{D1}$ is a α-maximal decision implication of K. Then,

(1) B_{C1} is a minimal generator of the concept $(B_{C1}^{C}, B_{C1}^{CC})$; and
(2) $B_{C1}^{CD}=B_{D1}$.

Proof.

(1) If B_{C1} is not a minimal general of $(B_{C1}^{C}, B_{C1}^{CC})$, then there must be a set $B_{C2}\subset B_{C1}$ such that $B_{C2}^{CC}=B_{C1}^{CC}$. By Theorem 1, we have $B_{C2}^{C}=B_{C1}^{C}$. Since $B_{C1}\rightarrow B_{D1}$ is a decision implication of K, by Theorem 2, we can confirm $B_{C1}^{C}\subseteq B_{D1}^{D}$ and hence $B_{C2}^{C}\subseteq B_{D1}^{D}$. From that, it follows that $B_{C2}\rightarrow B_{D1}$ is a decision implication, which contradict the α-maximal implication $B_{C1}\rightarrow B_{D1}$, since $B_{C2}\subset B_{C1}$. So B_{C1} is a minimal generator of $(B_{C1}^{C}, B_{C1}^{CC})$.

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(2) To start with, let us prove $B_{D1} = B_{D1}^{DD}$. Assume $B_{D1} \neq B_{D1}^{DD}$, then $B_{D1} \subset B_{D1}^{DD}$. Since $B_{C1} \rightarrow B_{D1}$ is a decision implication, then $B_{C1}^{C} \subseteq B_{D1}^{D} = B_{D1}^{DDD}$. Hence, $B_{C1} \rightarrow B_{D1}^{DD}$ holds in K contradictory with the condition that $B_{C1} \rightarrow B_{D1}$ is a α-maximal decision implication, since $B_{D1} \subset B_{D1}^{DD}$. Thus $B_{D1} = B_{D1}^{DD}$.

Next, we prove $B_{C1}^{CD} = B_{D1}^{DD}$. Since $B_{C1} \rightarrow B_{D1}$ is a decision implication of K, then $B_{C1}^{C} \subseteq B_{D1}^{D}$ and, by Theorem 1, $B_{C1}^{CD} \supseteq B_{D1}^{DD}$. If $B_{C1}^{CD} \neq B_{D1}^{DD}$, then $B_{C1}^{CD} \supset B_{D1}^{DD} = B_{D1}$, i.e. $B_{C1}^{CD} \supset B_{D1}$. By Theorem 6, $B_{C1} \rightarrow B_{C1}^{CD}$ is a decision implication contradictory with the fact that $B_{C1} \rightarrow B_{D1}$ is a α-maximal decision implication, since $B_{C1}^{CD} \supset B_{D1}$. So $B_{C1}^{CD} = B_{D1}^{DD}$.

Thus $B_{C1}^{CD} = B_{D1}^{DD} = B_{D1}$, which completes the proof. □

THEOREM 8. Let $K = (G, C \cup D, I_C \cup I_D)$ be a decision context and B_{C1} is a minimal generator of the condition sub-context $K_C = (G, C, I_C)$. Then, $B_{C1} \rightarrow B_{C1}^{CD}$ is a α-maximal decision implication of K if and only if, $B_{C1}^{CD} \neq B_{C2}^{CD}$ for any $B_{C2} \subset B_{C1}$.

Proof. "$\rightarrow$": Since $B_{C2} \subset B_{C1}$, then $B_{C1}^{CD} \subseteq B_{C2}^{CD}$. Assume $B_{C1}^{CD} = B_{C2}^{CD}$. Then, by Theorem 6, $B_{C2} \rightarrow B_{C2}^{CD}$ and hence $B_{C2} \rightarrow B_{C1}^{CD}$ hold. The later is contradictory with that $B_{C1} \rightarrow B_{C1}^{CD}$ is a α-maximal decision implication, since $B_{C2} \subset B_{C1}$. So $B_{C1}^{CD} \neq B_{C2}^{CD}$.

"$\leftarrow$": Assume that $B_{C1} \rightarrow B_{C1}^{CD}$ is not a α-maximal decision implication. Then, there are two cases to be considered.

(1) There exists $B_{C2} \subset B_{C1}$ such that $B_{C2} \rightarrow B_{C1}^{CD}$. By Theorem 2, we have $B_{C2}^{C} \subseteq B_{C1}^{CDD}$ and hence $B_{C2}^{CDD} \subseteq B_{C1}^{CDDDD} = B_{C1}^{CDD}$, i.e. $B_{C2}^{CDD} \subseteq B_{C1}^{CDD}$. On the other hand, since $B_{C2} \subset B_{C1}$, we have $B_{C1}^{C} \subseteq B_{C2}^{C}$ and hence $B_{C2}^{CD} \subseteq B_{C1}^{CD}$. So $B_{C1}^{CDD} \subseteq B_{C2}^{CDD}$. And we can conclude $B_{C1}^{CDD} = B_{C2}^{CDD}$, i.e. $B_{C1}^{CD} = B_{C2}^{CD}$ contradictory with the condition $B_{C1}^{CD} \neq B_{C2}^{CD}$. So this case does not hold.

(2) There exists $B_{C1}^{CD} \subset B_{D2}$ such that $B_{C1} \rightarrow B_{D2}$. In the case, by Theorem 2, we have $B_{C1}^{C} \subseteq B_{D2}^{D}$. Hence $B_{D2}^{DD} \subseteq B_{C1}^{CD}$. Since B_{D2}^{DD} is an intent of the decision sub-context $K_D = (G, D, I_D)$, we have $B_{D2} \subseteq B_{D2}^{DD}$ and hence $B_{D2} \subseteq B_{C1}^{CD}$ contradictory with $B_{C1}^{CD} \subset B_{D2}$. So this case does not hold too.

The cases (1) and (2) imply that $B_{C1} \rightarrow B_{C1}^{CD}$ is α-maximal. □

With Theorems 7 and 8, we conclude:

THEOREM 9. Let $K = (G, C \cup D, I_C \cup I_D)$ be a decision context. Then $B_{C1} \rightarrow B_{D1}$ is a α-maximal decision implication of K if and only if

(1) B_{C1} is a minimal generator of the condition sub-context K_C;
(2) the implication $B_{C1} \rightarrow B_{D1}$ has the form $B_{C1} \rightarrow B_{C1}^{CD}$ (i.e. $B_{D1} = B_{C1}^{CD}$); and
(3) if $B_{C2} \subset B_{C1}$, then $B_{C1}^{CD} \neq B_{C2}^{CD}$.

Proof. "$\rightarrow$": By Theorem 7, we have (1) and (2). In the case, the implication $B_{C1} \rightarrow B_{D1}$ has the form $B_{C1} \rightarrow B_{C1}^{CD}$, and by Theorem 8, the result (3) can be concluded.

"$\leftarrow$": By the conditions (1) and (2), we know that the implication $B_{C1} \rightarrow B_{D1}$ has the form $B_{C1} \rightarrow B_{C1}^{CD}$. Then, by Theorem 8 and the condition (3), the implication $B_{C1} \rightarrow B_{D1}$ is α-maximal. □

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Stumme *et al.* (2000) presented a new algorithm called Titanic for computing concept lattices whose generation procedure was first used in the Apriori algorithm for the specific case of frequent itemsets. The core of Titanic algorithm is to generate the minimal generators. Accordingly, the Titanic algorithm can be used for generating α-maximal decision implications, since the premises of α-maximal decision implications are minimal generators. According to Theorem 9, the following algorithm (Algorithm 1) can generate the system of all α-maximal decision implications, which is α-non-redundant and α-complete.

ALGORITHM 1.

(1) generate all minimal generators of $K_C = (G, C, I_C)$ by Titanic algorithm;
(2) for each minimal generator B_{C1}, check the correctness of $B_{C1}^{CD} \neq B_{C2}^{CD}$ where $B_{C2} \subset B_{C1}$. If $B_{C1}^{CD} \neq B_{C2}^{CD}$, generate the α-maximal decision implication $B_{C1} \rightarrow B_{C1}^{CD}$; and
(3) return all the α-maximal decision implications generated by step 2.

Note. as is known to all, the time complexity for generating minimal generators will be:

$$O\left(|M|\cdot\left(\mathrm{db}+\binom{|M|}{\left\lfloor\frac{|M|}{2}\right\rfloor}\cdot|G|\cdot|M|\right)\right)$$

where db is the access time of the formal context. And in our algorithm, we need to go over all subsets B_{C2} of B_{C1}, which at worse-case will take:

$$O\left(\binom{|M|}{\left\lfloor\frac{|M|}{2}\right\rfloor}^2\right).$$

So the algorithm has exponential time complexity relatively to $|M|$:

$$O\left(|M|\cdot\left(\mathrm{db}+\binom{|M|}{\left\lfloor\frac{|M|}{2}\right\rfloor}\cdot|G|\cdot|M|\right)+\binom{|M|}{\left\lfloor\frac{|M|}{2}\right\rfloor}^2\right).$$

That is not a good thing especially for larger experiments. So the algorithm only serve as a basis for many opportunities for further development.

6. An illustration

Let us illustrate Algorithm 1 by an example. A decision context $K = (G, C \cup D, I_C \cup I_D)$ is depicted by Table 1, where the objects $x_1, x_2, \ldots, x_8$ denote eight customers composing the set G, $a_1, a_2, \ldots, a_6$ denote six wares composing the set C and d_1, d_2 denote two wares composing the set D. In Table 1, if a customer x_i buys a ware a_j or d_k, we write "×" to express $x_i I_C a_j$ or $x_i I_D d_k$.

All decision implications with the minimal generators as their premises are listed in Table 2.

The set of all α-maximal decision implications generated by Algorithm 1 are listed in Table 3.

Obviously, the number of decision implications in Table 3 is quite smaller than that in Table 2. In fact, with the increasing of condition attributes, α-decision inference rule can

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Table 2. The decision implications.

$\emptyset \to \emptyset$	$\{a_1, a_4\} \to \{d_1\}$	$\{a_5, a_6\} \to \{d_1\}$
$\{a_1\} \to \{d_1\}$	$\{a_1, a_5\} \to \{d_1\}$	$\{a_1, a_2, a_3\} \to \{d_1, d_2\}$
$\{a_2\} \to \emptyset$	$\{a_2, a_3\} \to \{d_2\}$	$\{a_1, a_3, a_4\} \to \{d_1, d_2\}$
$\{a_3\} \to \emptyset$	$\{a_2, a_4\} \to \{d_1\}$	$\{a_1, a_3, a_5\} \to \{d_1, d_2\}$
$\{a_4\} \to \emptyset$	$\{a_2, a_6\} \to \emptyset$	$\{a_2, a_3, a_5\} \to \{d_1, d_2\}$
$\{a_5\} \to \emptyset$	$\{a_3, a_4\} \to \{d_2\}$	$\{a_3, a_4, a_5\} \to \{d_1, d_2\}$
$\{a_6\} \to \emptyset$	$\{a_3, a_5\} \to \emptyset$	$\{a_3, a_5, a_6\} \to \{d_1, d_2\}$
$\{a_1, a_2\} \to \{d_1\}$	$\{a_3, a_6\} \to \emptyset$	
$\{a_1, a_3\} \to \{d_1\}$	$\{a_4, a_5\} \to \{d_1\}$	

Table 3. The α-maximal decision implications.

$\emptyset \to \emptyset$	$\{a_4, a_5\} \to \{d_1\}$	$\{a_1, a_3, a_5\} \to \{d_1, d_2\}$
$\{a_1\} \to \{d_1\}$	$\{a_5, a_6\} \to \{d_1\}$	$\{a_2, a_3, a_5\} \to \{d_1, d_2\}$
$\{a_2, a_3\} \to \{d_2\}$	$\{a_1, a_2, a_3\} \to \{d_1, d_2\}$	$\{a_3, a_4, a_5\} \to \{d_1, d_2\}$
$\{a_2, a_4\} \to \{d_1\}$	$\{a_1, a_3, a_4\} \to \{d_1, d_2\}$	$\{a_3, a_5, a_6\} \to \{d_1, d_2\}$

reduce more the α-redundant decision implications. In addition, by Theorem 3, all the decision implications in Table 1 can be inferred from Table 3 with α-decision inference rule. For example, the implication $\{a_1, a_2\} \to \{d_1\}$ in Table 2. Since $\{a_1\} \subseteq \{a_1, a_2\}$ and $\{d_1\} \subseteq \{d_1\}$, then, it can be inferred from the implication $\{a_1\} \to \{d_1\}$ in Table 3. On the other hand, the implication $\{a_1\} \to \{d_1\}$ means that, those customers, who buy the ware a_1, also buy the ware $\{d_1\}$, while the implication $\{a_1, a_2\} \to \{d_1\}$ expresses that those buying a_1 and a_2 will buy the ware d_1. It is easy to see that the later is redundant to the former, since those buying a_1 and a_2 must buy the ware a_1. Another decision implication $\{a_1, a_2, a_3\} \to \{d_1\}$, although it is not in Table 2, can be inferred from the α-maximal decision implication $\{a_1, a_2, a_3\} \to \{d_1, d_2\}$.

7. Conclusion and further work

In the paper, we introduce the notions of decision context and decision implication to FCA. Since the number of decision implications in a decision context is an exponential increase to the scale of the decision context, an inference rule is proposed. The so-called α-decision inference rule is sound, intuitive, effective and easy to use. For users who want to obtain the decision implications, what to do is to generate the α-maximal decision implications rather than all the decision implications. On the other hand, our paper does not deal with inconsistent decision contexts and the relation between the structures of sub-contexts and the original context, which will be our further work.

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